# An invariant of link cobordisms from symplectic Khovanov homology

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**Abstract**

Symplectic Khovanov homology is an invariant of oriented links defined by Seidel and Smith [1] and conjectured to be isomorphic to Khovanov homology. I define morphisms (up to a global sign ambiguity) between symplectic Khovanov homology groups, corresponding to isotopy classes of smooth link cobordisms in $\mathbb{R}^3 \times [0,1]$ between a fixed pair of links. These morphisms define a functor from the category of links and such cobordisms to the category of abelian groups and group homomorphisms up to a sign ambiguity. This provides an extra structure for symplectic Khovanov homology and more generally an isotopy invariant of smooth surfaces in $\mathbb{R}^4$; a first step in proving the conjectured isomorphism of symplectic Khovanov homology and Khovanov homology. The maps themselves are defined using a generalisation of Seidel's relative invariant of exact Lefschetz fibrations [2] to exact Morse-Bott-Lefschetz fibrations with non-compact singular loci.

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# 1 Introduction

The Jones polynomial $V_{\mathfrak{L}}$, introduced in [3], is an invariant of oriented links $\mathfrak{L} \subset S^3$, motivated originally by a connection between the Yang-Baxter equation in statistical mechanics and the braid group. Importantly, $V_{\mathfrak{L}}$ satisfies skein relations:

$$t^{-1/2}V_{\asymp} + t^{3v/2}V_{)(} + t^{-1}V_{\times} = 0$$

$$t^{3v/2}V_{\asymp} + t^{1/2}V_{)(} + tV_{\times} = 0$$

The pictures in the skein relations above depict a region of a crossing diagram of three oriented links, which are identical elsewhere. $v$ is a term, arising as a signed count of certain crossings, which compensates for the fact that there is no canonical choice of orientation on one resolution of any crossing.

Together with the normalisation $V_{\text{unknot}} = 1$, this allows the Jones polynomial to be algorithmically computed in a straightforward, though computationally intensive, manner from a crossing diagram of a link. However, the geometric meaning of the Jones polynomial is still poorly understood. One hopes that an intrinsically geometric definition of the Jones polynomial would advance geometric applications to knot theory.

Khovanov [4, 5] defines a bigraded abelian group $\mathrm{KH}^{*,*}(\mathfrak{L})$, which categorifies the Jones polynomial. It is widely known as the *Khovanov homology* of $\mathfrak{L}$, although technically a cohomology theory. The skein relations for the Jones polynomial are replaced by long exact sequences for Khovanov homology (bi-degrees are indicated on the arrows):

$$\begin{tikzcd} & \mathrm{KH}^{*,*}(\times) \arrow[dr, "{(0,-1)}"] & \\ \mathrm{KH}^{*,*}()() \arrow[ur, "{(v+1,3v+2)}"] & & \mathrm{KH}^{*,*}(\asymp) \arrow[ll, "{(-v,-3v-1)}"'] \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{KH}^{*,*}(\times) \arrow[dr, "{(-v+1,-3v+2)}"] & \\ \mathrm{KH}^{*,*}()() \arrow[ur, "{(0,1)}"] & & \mathrm{KH}^{*,*}(\asymp) \arrow[ll, "{(v,3v+1)}"'] \end{tikzcd}$$

A consequence of this is that the Jones polynomial is recovered from a change of variables on the bigraded Poincaré polynomial of $\mathrm{KH}^{*,*}(\mathfrak{L})$.

$$V_{\mathfrak{L}} = \left[ \frac{\sum_{i,j} (-1)^i \dim(\mathrm{KH}^{i,j}(\mathfrak{L}) \otimes \mathbb{Q})}{q + q^{-1}} \right]_{q=t^{1/2}}$$

Khovanov homology is also defined in terms of algorithmically computable algebra related to crossing diagrams. Its other strengths include, in particular, that it fits into a topological

quantum field theory for knotted-surfaces in $\mathbb{R}^4$. This property was used by Rasmussen [6] to give a purely combinatorial proof of Milnor's conjecture on the smooth slice genus of torus knots. The same work can also be used to construct an exotic smooth structure on $\mathbb{R}^4$.

The topological field theory is described as follows. Let **Cob** be the category whose objects are oriented links in $S^3$ and whose morphisms $\mathfrak{L}_1 \longrightarrow \mathfrak{L}_1$ are isotopy classes of smooth link cobordisms from $\mathfrak{L}_1$ to $\mathfrak{L}_2$ in $S^3 \times [0,1]$. Let $\mathbf{Ab}_2^{\pm}$ be the category of finitely generated, bigraded abelian groups and bigraded homomorphisms defined only up to an overall sign ambiguity. It is shown [7] that there is a non-trivial functor from **Cob** to $\mathbf{Ab}_2^{\pm}$, which maps a link $\mathfrak{L}$ to the Khovanov homology $\mathrm{KH}^{*,*}(\mathfrak{L})$. To be more precise, it takes $\mathfrak{L}$ to the Khovanov homology of the crossing diagram created by projecting a $C^1$–small perturbation of $\mathfrak{L}$ orthogonally to $\{(0,0,z) \in \mathbb{R}^3\}$. One often phrases this result as "*Khovanov homology is functorial with respect to smooth link cobordims*".

*Symplectic Khovanov homology* $\mathrm{KH}^*_{\mathrm{symp}}(\mathfrak{L})$ is another invariant of oriented links, due to Seidel and Smith [1]. More recently, it has also been extended to an invariant of tangles by Rezazadegan [8]. For links, it is a singly-graded abelian group, defined using Lagrangian Floer cohomology in an auxiliary symplectic manifold $(\mathcal{Y}_{m,P}, \Omega)$, and Lagrangian submanifolds in $\mathcal{Y}_{m,P}$ derived from a presention of $\mathfrak{L}$ as a braid closure. Section 4 introduces symplectic Khovanov homology in more detail, as well as extending the definition to work with bridge diagrams of links. In doing so, we ignore orientations of the links, which amounts to replacing the absolute grading on symplectic Khovanov homology by a relative grading. For the purposes of this paper, we do not require a more precise understanding of the absolute grading. It suffices to know that the absolute grading always exists, given a choice of orientation.

Seidel and Smith conjecture that symplectic Khovanov homology is in fact the same as Khovanov homology. Since $\mathrm{KH}^*_{\mathrm{symp}}(\mathfrak{L})$ has only one grading, the conjecture is usually phrased as:

**Conjecture 1.** *There is a canonical isomorphism* $\mathrm{KH}^k_{\mathrm{symp}}(\mathfrak{L}) \cong \bigoplus_{i-j=k} \mathrm{KH}^{i,j}(\mathfrak{L})$

Here, one has also to be careful with the meaning of *canonical.* Section 5.6 defines the symplectic Khovanov homology of a crossing diagram up to canonical isomorphism, thus allowing one to be precise about the meaning of the above conjecture.

The main aim of this paper is to exhibit the same functoriality with respect to smooth link cobordisms for symplectic Khovanov homology as exists in the setting of Khovanov homology. Recent work of Rezazadegan [9] has independently exhibited similar homomorphisms between symplectic Khovanov homology groups (for tangles) corresponding to elementary cap, cup and saddle cobordisms. He also exhibits some important structure, relevant to Conjecture 1, but does not show that these homomorphisms give an invariant of link cobordisms.

**Theorem 1.1.** *Let* $\mathbf{Ab}_1^{\pm}$ *be the category of finitely generated, singly graded abelian groups and graded homomorphisms, defined only up to an overall sign ambiguity. There is a non-trivial functor from* **Cob** *to* $\mathbf{Ab}_1^{\pm}$*, which maps a link* $\mathfrak{L}$ *to the symplectic Khovanov homology* $\mathrm{KH}^*_{\mathrm{symp}}(\mathfrak{L})$.

In particular, some care is needed to define the symplectic Khovanov homology up to canonical isomorphism for a general link in $\mathbb{R}^3$. In fact, the same trick of defining symplectic Khovanov homology for crossing diagrams together with a small perturbation of the link works here too.

One can consider Theorem 1.1 as a first step in proving Conjecture 1, since Khovanov homology is defined in terms of elementary cobordisms between unlinked unions of unknots (and the homomorphisms corresponding to general cobordisms arise directly from these).

Combining the Theorem with a simple generalisation of Wehrheim and Woodward's exact sequence [10] (one has to deal with non-compactness issues similar to those in Section 2) one should already be able to exhibit skein exact triangles for symplectic Khovanov homology. In particular, Theorem 1.1 should imply the existence of a spectral sequence with $\mathbb{Z}/2$–coefficients from Khovanov homology to symplectic Khovanov homology (cf. [11]).

I also prove the less surprising, though previously unproven, result that the symplectic Khovanov homology of an unlinked union of links splits as a tensor product at the level of chain complexes, so is described by the Künneth isomorphism on the level of homology. This result was previously only shown in the case that one of the two links was an unknot [1, Proposition 56].

**Theorem 1.2.** *Let $\mathfrak{L}$ be an unlinked union of links $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Then $\mathrm{KH}^*_{\mathrm{symp}}(\mathfrak{L})$ is the Künneth product of $\mathrm{KH}^*_{\mathrm{symp}}(\mathfrak{L}_1)$ and $\mathrm{KH}^*_{\mathrm{symp}}(\mathfrak{L}_2)$ (i.e. tensor product at the level of chain complexes) up to an overall grading shift.*

The underlying motivation for the proofs of both results is in the geometry of configuration spaces $\mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$ and $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$. In the construction of symplectic Khovanov homology, one uses the following correspondences to present links in terms of these configuration spaces:

- A braid joining a configuration $P \in \mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$ of points in the plane to another configuration $Q$ corresponds to a path $P$ to $Q$ in the configuration space.
- Let $\gamma : [0,1] \longrightarrow \overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ be a *vanishing path*; that is, a smooth path hitting $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C}) \setminus \mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$ transversely, only at $\gamma(0) = (0, 0, \mu_1, \ldots, \mu_{2n-2})$ such that the $\mu_i$ are pairwise distinct. Then $\gamma$ corresponds to a $(2n-2, 2n)$ tangle between configurations $\gamma(0)$ and $\gamma(1)$.

Suppose we have a map $u : \overline{\mathbb{D}} \longrightarrow \overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ intersecting $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C}) \setminus \mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$ transversely only on $\mathbb{D}$ and only in configurations $(0, 0, \mu_1, \ldots, \mu_{2n-2})$, such that the $\mu_i$ are pairwise distinct. Then from this one can construct a smooth braid cobordism from the trivial braid at $u(1)$ to the braid described by $u(\partial\overline{\mathbb{D}})$. In fact, braid cobordisms up to isotopy, correspond to isotopy classes of such maps. For details on this correspondence see Section 5.

With the further assumption that $u$ is a holomorphic embedding near where it intersects $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C}) \setminus \mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$, there is a natural construction of a singular symplectic fibration over $\overline{\mathbb{D}}$. These have fibres over $\pm 1$ which are the the auxiliary symplectic manifolds $\mathcal{Y}_{n,u(\pm 1)}$, in which $\mathrm{KH}^*_{\mathrm{symp}}$ is defined as a Floer cohomology group. In a manner motivated by Seidel's relative invariant of exact Lefschetz fibrations [2], one would then like to define morphisms between symplectic Khovanov homology groups by counting holomorphic sections of these fibrations.

In fact, the above describes essentially the method of this paper. The fibrations are all of a particularly nice form (exact Morse-Bott-Lefschetz fibrations) and relative invariants can be defined in an analogous manner. However, essential non-compactness issues (arising from non-compactness of the singular loci) cause problems for convexity, gluing and even defining symplectic parallel transport in these fibrations.

Section 2 develops general tools for studying relative invariants in these fibrations. In particular, it is also shown in general how to construct exact Morse-Bott-Lefschetz fibrations together with relative invariants from maps of surfaces into singular holomorphic fibrations of Stein manifolds. Section 3 then develops some specific tools for the calculation of Floer cohomology and relative invariants of symplectic associated bundles necessary for proof of Theorem 1.1 given in the later sections.

## 2 Floer cohomology and singular symplectic fibrations

Given a pair $L$, $L'$ of closed, connected Lagrangian submanifolds of a symplectic manifold $M$ satisfying certain conditions, one can define a relatively graded abelian group $HF(L, L')$. This is the Floer cohomology, or in our case, the "Lagrangian intersection Floer cohomology". The cohomology is obtained from a chain complex made of formal sums (over $\mathbb{Z}$) of transverse intersections of the Lagrangians and a differential defined by counting holomorphic strips with boundary on the Lagrangians which interpolate between intersections.

**Remark 2.1.** An example of conditions in which the Floer cohomology is defined is given by the following:

- $M$ is a Kähler manifold on which the symplectic form is exact and the underlying complex structure makes $M$ a Stein manifold
- $c_1(M) = 0$ and $H^1(M) = 0$
- $H_1(L) = H_1(L') = 0$
- $w_2(L) = w_2(L') = 0$ (equivalently $L$, $L'$ are spin)

These conditions are satisfied where Floer cohomology is used for the definition of symplectic Khovanov homology in [1]. Unless otherwise mentioned these are the conditions under which we will use Floer cohomology in this paper.

Floer cohomology is defined up to canonical isomorphism, even when each Lagrangian is specified only up to compactly supported Hamiltonian isotopy. These isotopies and canonical isomorphisms are used to define the cohomology even when the Lagrangians do not intersect transversely.

**Remark 2.2.** The condition that $L$, $L'$ be spin is necessary only to use $\mathbb{Z}$ coefficients for $HF(L, L')$. Without it Floer cohomology is still defined with $\mathbb{Z}/2$ coefficients (provided $L, L'$ are orientable.

$L$, $L'$ being spin implies the orientability of the moduli spaces of holomorphic strips (see Lemma 22.11 of [12]), the counting of which defines the differential. The orientation gives a consistent choice of signs for this counting process.

A symplectic vector bundle (cf. [13]) is a vector bundle $E \longrightarrow B$ with a smooth choice of skew symmetric bilinear form on each fibre (i.e. a section $\Omega$ of $\Lambda^2 E$), which is non-degenerate on each fibre. This is the local (first order) model for a symplectic fibration (where $\Omega$ is instead a closed 2–form on the total space).

To be more precise, an *exact symplectic fibration* is a manifold E with corners and a smooth fibration $\pi\colon E \longrightarrow B$ equipped with an exact 2–form $\Omega = d\Theta$ on E whose restriction to fibres of E is a symplectic form. We shall also require that the corners of E are precisely the boundary points of the fibres over $\partial B$.

Non-degeneracy of $\Omega$ on the vertical tangent spaces $TE^v = \ker D\pi$ means that we can define horizontal tangents to be $TE^h = (\ker D\pi)^{\perp_\Omega}$. This defines the *symplectic connection* and *symplectic parallel transport* over any path $\gamma$ in the base. As long as points do not flow under symplectic parallel transport off of the boundary of $E$, the symplectic parallel transport defines maps between the fibres over the start and end points of $\gamma$. These maps are symplectomorphisms between the fibres. Isotopic paths in the base yield parallel transport maps which differ by exact symplectomorphisms.

In [2] these fibrations are generalised to *exact Lefschetz fibrations* (over surfaces), by allowing complex non-degenerate singularities of $\pi$. The monodromy by parallel transport once anticlockwise around such a singular value in the base is then a Dehn twist $\sigma$ in the Lagrangian vanishing cycle associated to the singular point. Take an exact Lefschetz fibration over the infinite strip $\mathbb{R} \times [0,1]$ which has trivialised symplectic parallel transport over the ends (giving well defined fibres at $\pm\infty$). One assigns to the fibre at $+\infty$ a pair of exact Lagrangian submanifolds $L^0_{+\infty}$, $L^1_{+\infty}$. Extending these by symplectic parallel transport over the boundaries $\{0\} \times \mathbb{R}$, $\{1\} \times \mathbb{R}$ respectively, Seidel defines a map from the Floer cohomology in the fibre at $-\infty$ to that in the fibre at $+\infty$.

$$HF(L^{top}_{-\infty} = \sigma L^{top}_{+\infty}, L^{bottom}_{-\infty}) \longrightarrow HF(L^{top}_{+\infty}, L^{bottom}_{+\infty})$$

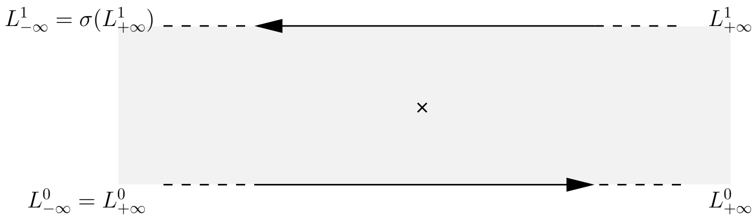


Figure 1: A basic Lefschetz fibration over the infinite strip with one singular fibre. The orientation indicated on the boundary is such as to make the total monodromy a single positive Dehn twist $\sigma$ in the vanishing cycle. Furthermore the fibres at $\pm\infty$ have been identified by symplectic parallel transport along the lower boundary, so the monodromy occurs entirely on the upper boundary.

An important concept for calculations with Lefschetz fibrations is that of *symplectic curvature*.

**Definition 2.3.** Let $(E, \pi)$ be a symplectic fibration over a surface $B$. The *symplectic curvature* $F$ is a 2–form on $B$ with values in functions on the fibres. Given vectors $Z_1, Z_2 \in TB$ we define $F(Z_1, Z_2) := \Omega(Z_1^h, Z_2^h)$ where $Z_1^h, Z_2^h$ are the lifts of $Z_1, Z_2$ to $TE^h$.

This is not quite the same as the curvature of the symplectic connection. In fact adding to $\Omega$ the pullback $\pi^*(f dx \wedge dy)$ of a 2–form $f dx \wedge dy$ in the base $B$ does not change the connection, but adds $\pi^* f \otimes dx \wedge dy$ to the curvature.

Suppose the symplectic curvature is $H \otimes dx \wedge dy$, then the curvature of the symplectic connection is $X_H \otimes dx \wedge dy$, where $X_H$ is the fibrewise Hamiltonian vector field of $H$. This is not changed by adding to $\Omega$ the pullback of a 2–form from the base, since that changes $H$ only by a fibrewise constant function.

To see this one applies Moser's Lemma to a smooth trivialisation of $E$ near a point. This provides a new trivialisation which is respected by $\Omega|_{(TE^v)^{\otimes 2}}$. On this trivialisation the connection is expressed as $d + A$, where $A$ is of a nice form. Namely, $A = -\sum_i X_{H_i} \otimes dx_i$ where $X_{H_i}$ are again the fibrewise Hamiltonian vector fields of functions $H_i$. Computing the curvature of $d + A$ expresses it, as described above, as the 'Hamiltonian vector field of the symplectic curvature'.

**Definition 2.4.** Suppose we have a volume form $\omega$ on $B$, then the symplectic curvature $F$ can be written uniquely as $f \otimes \omega$ for some $f \in C^\infty(E)$. In this case we shall refer to the symplectic curvature simply as $f$. We say the curvature is positive, negative, zero, etc. if $f$ is positive,

negative, zero, etc. respectively. Furthermore, these properties are well-defined in the case that we have chosen only an orientation on $B$, not a volume form. This is, in particular, the case when $B$ carries a complex structure.

**Remark 2.5.** Symplectic curvature being fibrewise locally constant implies that the actual curvature is zero. This gives local trivialisations respected by $\Omega|_{(TE^v)^{\otimes 2}}$ and by the connection. However, positivity of symplectic curvature is not so simply expressed in terms of actual curvature. The reason we are interested in the sign of symplectic curvature is that it is useful in bounding the action $A(u) := \int_B u^*\Omega$ of sections $u$ of $E$. In particular, horizontal sections through regions of zero symplectic curvature will necessarily have zero action. More detail is given on this in Sections 2.3 and 2.4 of this paper.

## 2.1 Exact Morse-Bott-Lefschetz fibrations

Working with $\mathrm{KH}_{\mathrm{symp}}$, we have a natural construction of singular symplectic fibrations from braid cobordisms. The singularities that arise in this construction have a slightly more general form than those of the exact Lefschetz fibrations, considered in [2]. In this section, I describe the corresponding construction for exact Morse-Bott-Lefschetz (MBL) fibrations over surfaces.

**Definition 2.6.** An exact *MBL–fibration* is a collection $(E, \pi, \Omega, \Theta, J_0, B, j)$ such that:

(1) $E$ is a smooth, not necessarily compact, manifold with boundary $\partial E$.

(2) $B$ is a Riemann surface with complex structure $j$, homeomorphic to $\overline{\mathbb{D}}$ with finitely many boundary points removed.

(3) $\pi\colon E \longrightarrow B$ is a smooth map with $\partial E = \pi^{-1}(\partial B)$ and such that $\pi|_{\partial E}\colon \partial E \longrightarrow \partial B$ is a smooth fibre bundle.

(4) $\Omega = d\Theta$ is an exact 2–form on $E$, non-degenerate on $TE^v := \ker D\pi$ at every point in $E$.

(5) $\pi$ has finitely many critical values, all in the interior of $B$.

(6) $J_0$ is an almost complex structure defined on some subset of $E$ which contains a neighbourhood of the set $\mathrm{Crit}(\pi)$ of critical points and the complement $U$ of some fibrewise compact subset of $E$.

(7) $\pi$ is $(J_0, j)$–holomorphic and $\Omega(., J_0.)|_{(TE^v)^{\otimes 2}}$ is everywhere symmetric and positive definite (where $J_0$ is defined).

(8) $J_0$ preserves $TE^h$ on $U$.

(9) $\Omega$ is a Kähler form for $J_0$ on some open neighbourhood of $\mathrm{Crit}(\pi)$.

(10) $\mathrm{Crit}(\pi)$ is smooth and the complex Hessian of $\pi$ is non-degenerate on complex complements of $T\,\mathrm{Crit}(\pi)$ in $TE$.

Seidel's exact Lefschetz fibrations have boundary in the fibre direction near which there is a trivialisation of $E$ compatible with $\Omega$ and $\Theta$ (c.f. [2]). One cannot expect such trivialisations at boundaries to exist for exact MBL-fibrations, since the singular locus can escape to infinity in a fibre. For this reason, we don't require there to be trivialisations. We compensate for this by taking significantly more care with convexity and gluing of exact MBL-fibrations. This is the main difficult content of Section 2.2.

We will consider exact MBL-fibrations with bases $B$ which are of a particular form. Namely $B$ should be a Riemann surface with finite sets $I^{\pm}$ of *ends* (see below), not both empty. The ends may be of two forms:

**Definition 2.7** (Striplike ends (cf. [2]))**.**
A *striplike end* $e \in I^{\pm}$ of a surface $B$ is a proper holomorphic embedding

$$\gamma_e \colon \mathbb{R}^{\pm} \times [0,1] \longrightarrow B$$

(with the standard complex structure on $\mathbb{R}^{\pm} \times [0,1] \subset \mathbb{C}$) such that $\gamma_e^{-1}(\partial B) = \mathbb{R}^{\pm} \times \{0,1\}$.

An exact MBL–fibration is *trivial over the striplike end* $e$ if over the image of $\gamma_e$ it is non singular and isomorphic as an exact symplectic fibration to $\mathbb{R}^{\pm} \times [0;1] \times E_z$ for some fibre $E_z$. Here one takes $\Omega$ and $\Theta$ pulled back by the projection to $E_z$, and $J_0$ split as the sum of an almost complex structure on the $E_z$–factor and the standard almost complex structure on the $\mathbb{R}^{\pm} \times [0,1]$–factor.

**Definition 2.8.** A *boundary marked point* $z \in \partial B$ together with a proper holomorphic embedding

$$\gamma_e \colon \mathbb{R}^{\pm} \times [0,1] \longrightarrow B \setminus z$$

such that $\gamma_e^{-1}(\partial B) = \mathbb{R}^{\pm} \times \{0,1\}$ and $\gamma_e(x,t) \longrightarrow z$ as $x \longrightarrow \pm\infty$ may also be considered an end. Exact MBL-fibrations are not required to be trivial over these ends.

Ends given by boundary marked points can be viewed as striplike ends without the trivialisation and striplike ends can be completed, by adding a single *fibre at infinity*, to give boundary marked points. Switching between these two settings will be important later on.

**Definition 2.9.** By the *fibre at an end* $e$ of $B$ we mean:

- the fibre over the boundary marked point
- the *fibre at infinity* of a striplike end

We require also, that the ends of $B$ be pairwise disjoint and that the complement of the ends (i.e. of the images of the $\gamma_e$ and any boundary marked points) be compact. This means in particular that the boundary of $B$ with boundary marked points removed decomposes into as many open intervals as there are ends.

The benefit of striplike ends (with accompanying trivialisations) is that it is easy to *compose* exact MBL–fibrations at striplike ends. Namely, one forms the composite by gluing oppositely oriented, but otherwise identical trivialisations of two separate exact MBL–fibrations together.

In contrast, the benefit of boundary marked points is twofold. They arise more naturally (see Section 2.2) and holomorphic convexity in the fibre direction is easier to attain.

We now define what we mean by *exact Lagrangian boundary conditions* for an exact MBL-fibration $(E,\pi)$ over a surface $B$ with ends.

**Definition 2.10.** Let $P$ be the set of boundary marked points of $B$. An *exact Lagrangian boundary condition* on $(E,\pi)$ is a subbundle $Q$ of $E$ over $\partial B \setminus P$ together with a function $K_Q \colon Q \longrightarrow \mathbb{R}$ such that:

1. $\Omega|_Q = 0$
2. for any $z \in \partial B$ the restriction $(Q_z, K_Q|_{Q_z})$ is a closed, connected exact Lagrangian submanifold of $E_z$ (i.e. a Lagrangian submanifold such that also $d(K_Q|_{Q_z}) = \Theta|_{Q_z}$).

3. $(Q_z, K_Q|_{E_z})$ extends smoothly, along each component of $\partial B \setminus P$, to the fibres over boundary marked points. This extension is allowed to depend on the side from which one approaches a boundary marked point.

4. $(Q_z, K_Q|_{E_z})$ is constant w.r.t. trivialisations over the striplike ends

Given an exact MBL–fibration over a surface with striplike ends, one can construct an exact MBL–fibration over a surface with boundary marked points. Namely, the base can be compactified by adding a single point *at infinity* at each end. One then adds to the total space the fibres at infinity.

Condition (1) implies that $Q$ is preserved by symplectic parallel transport over $\partial B$ and that $\Theta|_Q = dK_Q + \pi^* \kappa_Q$ for some $\kappa_Q \in \Omega^1(\partial B)$ (cf. [2] Lemma 1.3). Condition (2) and triviality of the striplike ends gives $\kappa_Q = 0$ there. In fact $\kappa_Q = 0$ whenever symplectic parallel transport preserves $K_Q$.

$Q$ specifies a pair of Lagrangian submanifolds in the fibre over each marked point and in each fibre at infinity (i.e. 'in the fibre at each end'). We will refer to $Q$ as *transverse*, if these pairs of Lagrangians are each transverse.

**Remark 2.11.** Assume we are given a choice of exact Lagrangian submanifold $(L, K_L)$ in the fibre at infinity or fibre over a boundary marked point at one end of each edge of $\partial B$. Then either symplectic parallel transport maps restricted to one of these Lagrangians fail to be defined over the entire edge on which it lies, or else they are defined and the condition $\kappa_Q = 0$ uniquely specifies a Lagrangian boundary condition. This is the manner in which all exact Lagrangian boundary conditions in this paper are constructed.

We will be interested in counting compact moduli spaces of holomorphic sections with boundary in $Q$, so it makes sense to require holomorphic convexity of a neigbourhood in $E$ containing $Q$ as follows:

**Definition 2.12.** An *enclosed exact Lagrangian boundary condition* $(Q, \rho, R)$ is an exact Lagrangian boundary condition $Q$, together with a smooth map $\rho\colon\ E \longrightarrow \mathbb{R}^{\geq 0}$ and $R \in \mathbb{R}$, such that:

(i) $\rho$ splits w.r.t. the trivialisations over all striplike ends

(ii) $\rho^{-1}[0, R]$ is fibrewise compact and $\rho^{-1}[0, R)$ contains $Q$

(iii) $\exists \varepsilon > 0$ such that on $\rho^{-1}(R - \varepsilon, R)$

- $J_0$ is defined
- $\rho$ is subharmonic w.r.t. $J_0$

We refer to the pair $(\rho, R)$ as an *enclosure.*

Suppose we extend $J_0$ to an almost complex structure on $E$ which makes the projection $\pi$ holomorphic. Then subharmonicity of $\rho$ gives a maximum principle for holomorphic sections of $E$. Namely, they cannot have a maximum of $\rho$ in the range $(R - \epsilon, R)$. This ensures that families of holomorphic sections which are confined to $\rho^{-1}[0, R - \epsilon)$ cannot degenerate to sections which escape this region. The choice of enclosure is important, since in general there will exist holomorphic sections which leave the region $\rho^{-1}[0, R)$.

It will often be necessary to change enclosures other than just by isotopy through enclosures. For this, we will need a notion of equivalence of enclosures.

**Definition 2.13.** We say that two enclosures $(\rho, R)$ and $(\sigma, S)$, for a given exact Lagrangian boundary condition $Q$, are *equivalent* if for some $R' < R$ we have:

- $\rho^{-1}[0, R') \subset \sigma^{-1}[0, S) \subset \rho^{-1}[0, R)$
- $\rho$ is subharmonic on $\rho^{-1}[R', R]$

or also if they can be related by a sequence of such comparisons, either way round. For simplicity, we require that $J_0$ is fixed throughout these comparisons.

In particular this makes enclosures related by isotopy of enclosures equivalent.

As defined earlier, an exact MBL–fibration carries a two form which is not necessarily symplectic on the total space. It's important to observe that this is more flexible than requiring $\Omega$ to be an exact symplectic structure on the total space (within a particular enclosure), but it is no weaker for our purposes.

**Lemma 2.14.** *Let $(E, \pi, \Omega, \Theta, J_0, B, j)$ be an exact MBL–fibration. Given any enclosure $(\rho, R)$, there is a canonical choice of isotopy class of exact symplectic form $\Omega'$ on $\rho^{-1}[0, R]$ with:*

- *the same restriction to $TE^v$ as $\Omega$*
- *the same symplectic connection as $\Omega$*

*Proof.* Let $\omega$ be any exact volume form on $B$ compatible with $j$ (which in 2 dimensions means only that it induces the correct orientation). $\Omega$ is non-degenerate at $z \in E \setminus \mathrm{Crit}(\pi)$ iff the symplectic curvature $f(z)$ is non-zero. Furthermore, $\Omega$ is non-degenerate at all $z \in \mathrm{Crit}(\pi)$ by definition.

$\Omega$ is Kähler on some open neighbourhood $V$ of $\mathrm{Crit}(\pi)$, so $f$ is strictly positive on $V \setminus \mathrm{Crit}(\pi)$. The region $(E \setminus V) \cap \rho^{-1}[0, R)$ is fibrewise compact and becomes compact when one extends to the fibres at infinity. $f$ is smooth away from $V$, so is bounded below. $\Omega' := k\pi^*\omega + \Omega$ has symplectic curvature $f + k$ and satisfies the necessary axioms for an exact MBL–fibration so for large enough $k \in \mathbb{R}$ it is as required in the lemma. □

**Remark 2.15.** Lemma 2.14 allows one to perform standard holomorphic disc counting constructions within the enclosure $(\rho, R)$ to get invariants of exact Lagrangian boundary conditions up to Lagrangian isotopy in a similar manner to the way one defines Floer cohomology in exact symplectic manifolds. See Section 2.3 for more details.

## 2.2 Constructing MBL–fibrations from Stein manifolds

Let $E$ be a Stein manifold with plurisubharmonic function $\rho \geq 0$ and exact Kähler form $\Omega := d\Theta := -d(d\rho \circ i)$. Suppose we have a singular holomorphic fibration $\pi\colon E \longrightarrow N$ over a complex manifold $N$ and a smooth map $u\colon B \longrightarrow N$, for some simply connected Riemann surface $B$ with striplike ends or marked points on the boundary. When the singularities of $u^*E$ are of Morse-Bott-Lefschetz type and $u$ is holomorphic near singular points, one can view $u^*E$ naturally as an exact MBL–fibration.

This section gives a more detailed construction of exact MBL–fibrations in the manner described above. Most of the content deals with the problems of convexity (for defining enclosed Lagrangian boundary conditions consistently) and composition of fibrations (by *gluing* trivialisations over striplike ends). Without these trivialisations, the gluing construction for composing the relative invariants (cf. Section 2.3) would be difficult.

**Definition 2.16.** We say a singular value $p \in N$ of $\pi$ is MBL if there is a neighbourhood of any point in the singular locus $\mathrm{Crit}(\pi, p)$ fitting into the following commutative diagram

$$\begin{tikzcd} E \arrow[d, "\pi"'] & & \mathrm{Crit}(\pi,p) \times \mathbb{C}^k \times \mathbb{C}^n \arrow[ll, "\text{local near } \mathrm{Crit}(\pi,p)"'] \arrow[d, "(\mathbf{x},\mathbf{y},\mathbf{z}) \mapsto (\sum y_i^2, \mathbf{z})"] \\ N & & \mathbb{C} \times \mathbb{C}^n \arrow[ll, "\text{local biholomorphism near } (0,\mathbf{0})"'] \end{tikzcd}$$

Here the top map is a local embedding of exact Kähler manifolds defined on a neighbourhood of $\mathrm{Crit}(\pi, p)$. However, so long as it is holomorphic, one can deform the exact Kähler structure on $E$ to make it an embedding of exact Kähler manifolds.

We shall denote the set of such critical values by $\mathrm{MBL}(\pi)$. It is a submanifold of codimension 2. The open set of regular values we denote by $N^{reg}$.

This definition can equivalently be expressed in terms of smoothness of critical loci in $E$ and $N$ and non-degeneracy of the Hessian of $\pi$ on complements to $T\,\mathrm{Crit}(\pi) \cap \ker(D\pi)$ within $\ker(D\pi)$.

**Definition 2.17.** Let $(B, j)$ be a simply connected Riemann surface $(B, j)$ with striplike ends or marked points on its boundary. Let $u\colon B \longrightarrow N^{reg} \cup \mathrm{MBL}(\pi)$ be a smooth map with the properties that:

- $u$ is a constant map on each of the striplike ends
- $u(\partial B) \subset N^{reg}$
- $u$ is transverse to $\mathrm{MBL}(\pi)$ and holomorphic near it.

We call such a map *admissible*. We shall refer to $u^{-1}(\mathrm{MBL}(\pi)) \subset (B, j)$ as the *singular values*, since these are the singular values of the pullback fibration $u^*E$.

Given an admissible map $u$, we consider the pullback fibration $\pi\colon u^*E \longrightarrow B$ equipped with $u^*\Omega$, $u^*\Theta$, $u^*\rho$. There is also a natural choice of almost complex structure $\tilde{J}$ which agrees with $u^*J$ where $u$ is an immersion. Namely, one takes $u^*J$ on vertical tangents and $i$ on horizontal tangents with respect to the symplectic connection induced by $u^*\Omega$ (and $u^*J$ where the fibration is singular). These choices make $(u^*M, \pi, u^*\Omega, u^*\Theta, \tilde{J}, B, j)$ an exact MBL–fibration.

Furthermore, the isotopy class of $u$ through admissible maps defines an isotopy class of exact MBL–fibration.

**Lemma 2.18.** *Given $u$, $B$ and $j$ as above up to smooth homotopy of $u$ and deformation of $j$ we have an exact MBL–fibration defined up to smooth deformation of the parameters.*

The rest of this section deals with the deformations needed to construct Lagrangian boundary conditions and then ensure holomorphic convexity of a surrounding region (thus making an enclosed Lagrangian boundary condition). The approach is to approximate $B$ by a tree of embedded holomorphic discs connected at marked points and solve the same problem for embedded holomorphic discs.

First we consider the model case where $u$ is a holomorphic embedding and $B$ is the closed unit disc $\overline{\mathbb{D}}$ with finitely many (but at least one) marked points on the boundary. In this case the exhausting, plurisubharmonic function $\rho\colon E \longrightarrow [0, \infty)$ pulls back to a fibrewise-exhausting, plurisubharmonic function on $u^*E$.

**Lemma 2.19.** *Let $(E, \pi, \Omega, \Theta, J_0, \overline{\mathbb{D}}, j)$ be an exact MBL–fibration over the closed unit disc. Assume further that we have $\rho\colon E \longrightarrow [0,\infty)$ exhausting (fibrewise), and plurisubharmonic where $J_0$ is defined, such that $d\rho = \Theta \circ J_0$.*

*For any $l \in \mathbb{R}$, one can deform $\Theta$ (without changing the restriction of $\Omega$ to fibres) inside some level set $\rho_{max}$ of $\rho$ such that symplectic parallel transport flow lines over paths of length at most $l$ in $\partial\overline{\mathbb{D}}$ do not leave $\rho^{-1}[0, \rho_{max}]$.*

*The deformation occurs only over a small open neighbourhood of $\partial\overline{\mathbb{D}}$ and is well defined up to isotopy through such deformations of $\Theta$. Furthermore, the deformation may be chosen to have support disjoint from any particular compact set.*

*Proof.* Let $A$ be a small annular neighbourhood of $\partial\overline{\mathbb{D}}$ not containing any critical values of $\pi$. Let $\rho_0$ be large enough such that for all $z \in A$ all critical values of $\rho|_A$ are less than $\rho$. This implies that $\rho|^{-1}_{\pi^{-1}(A)}(\rho_0)$ is a smooth fibration over $A$ with compact fibre $C$ and hence carries a flat connection, well defined up to isotopy. Choose one. It gives a trivialisation over any small open neighbourhood $U \subset A$ in the base of the form $\mathrm{proj}_U\colon C \times U \longrightarrow U$. Extending this in the positive time direction by the fibrewise Liouville flow we have a trivialisation $\mathrm{proj}_U\colon C \times [0,\infty) \times U \longrightarrow U$.

On any fibre $E_z$ the form $\Theta$ restricts to $C \times \{0\} \times z$ as a contact form $\Theta_{0,z}$ on $C$ and restricts to the whole fibre as $e^y\Theta_{0,z}$ (here y is the coordinate on $[0,\infty)$). We define $\Theta_C$ on the trivialisation $C \times [0,\infty) \times U$ to have this same restriction to fibres and to vanish on $TU$. In particular $\Theta_C|_{E_z} = \Theta|_{E_z}$.

Let $R$ be the Reeb vector field on $C$ for the contact form $\Theta_{0,z}$. We can split $TE$ in the trivialised region as $\mathbb{R}R \oplus \ker(\Theta_{0,z}) \oplus \mathbb{R}\frac{\partial}{\partial y} \oplus TU$. Given a vector $H$ in $TU$ the symplectic parallel transport vector w.r.t. the 2–form $d(e^y\Theta_C)$ over it is of the form $w_R R + W_{con} + w_y\frac{\partial}{\partial y} + H$ in that splitting. It has the defining property that for any $v_R R + V_{con} + v_y\frac{\partial}{\partial y}$ we have:

$$\begin{aligned}
0 &= d(e^y\Theta_C)\left(w_R R + W_{con} + w_y\frac{\partial}{\partial y} + H, v_R R + V_{con} + v_y\frac{\partial}{\partial y}\right)\\
&= (e^y d\Theta_C + e^y dy \wedge \Theta_C)\left(w_R R + W_{con} + w_y\frac{\partial}{\partial y} + H, v_R R + V_{con} + v_y\frac{\partial}{\partial y}\right)\\
&= e^y\left[-H(\Theta_C)(v_R R + V_{con}) + d(\Theta_{0,z})(V_{con}, W_{con}) + v_y w_R - v_R w_y\right]
\end{aligned}$$

Setting $v_R = 1$, $V_{con}, v_y = 0$ gives $w_y = -H(\Theta_C)(R)$ which by compactness has a finite maximum over $A$. i.e. the velocity of this symplectic parallel transport in the $y$–direction is bounded over compact subsets of the base. This controls the symplectic parallel transport flow lines in large enough regions, so in particular proves the Lemma for any deformed $\Theta$ which equals $e^y\Theta_C$ on $\rho|^{-1}_{\pi^{-1}(A)}[\rho_0 + 1, K - 1)$ for large enough $K$.

Now we define a deformation $\tilde{\Theta} = g\Theta + (1-g)\Theta_C$ with a bump function $g$ identically equal to 1 on $\rho|^{-1}_{\pi^{-1}(A)}[\rho_0 + 1, K - 1)$ and zero on the complement of $\rho|^{-1}_{\pi^{-1}(A)}[\rho_0, K)$. This is the required deformation of $\Theta$ to prove the Lemma.

□

**Remark 2.20.** Suppose that $\mathrm{Crit}(\pi)$ is compact. For example this is the case when the singularities are actually of Lefschetz type. Then, by the same technique, one can contain the symplectic parallel transport over paths of length at most $l$ in $\overline{\mathbb{D}}$, not just in the boundary.

**Remark 2.21.** Suppose we are really just interested in defining symplectic parallel transport maps over a path $\gamma\colon [0,1] \longrightarrow N^{reg}$, then one can run the same argument in the pullback

fibration $\gamma^* E$. This gives symplectomorphisms $E_{\gamma(0)} \longrightarrow E_{\gamma(1)}$ defined on any compact subset of $E_{\gamma(0)}$ and well defined up to isotopy within the class of symplectic embeddings (or also inclusion, should one enlarge the choice of compact subset).

The Lemma above allows us to define Lagrangian boundary conditions on such a fibration simply by specifying a Lagrangian in a single fibre on each interval of $\partial\overline{\mathbb{D}}$ and extending to the rest of the interval by symplectic parallel transport. Call this Lagrangian boundary condition $Q$. Furthermore, the region of deformation is contained within a finite level set of $\rho$, so for all large enough $R \in \mathbb{R}^{\geq 0}$ the collection $(Q, \rho, R)$ is an enclosed Lagrangian boundary condition. If the Lagrangians in the construction are chosen to be exact, then $Q$ is also exact.

Given a surface with striplike ends mapping to $\overline{\mathbb{D}}$ with edges mapping monotonically to $\partial\overline{\mathbb{D}}$, symplectic parallel transport respects the pullback. Hence, we can similarly control symplectic parallel transport of compact Lagrangian boundary conditions specified in the fibre at infinity over one end of each edge. However, it is not so easy to show these are enclosed.

We will now construct enclosures containing these Lagrangian boundary conditions in a model case where we have deformed the previous fibration over $\overline{\mathbb{D}}$ to have a base with striplike ends. The enclosures constructed will be defined as deformations of an enclosure $(\rho, R)$ performed together with the deformation which forms the striplike ends. Furthermore, the enclosures will be compatible with the trivialisations of $E$ over the striplike ends, so will be compatible with the construction of gluing fibrations over striplike ends.

Label the marked points $\{z_1, \ldots, z_n\} \subset \partial\overline{\mathbb{D}}$. A neighbourhood of each of these marked points becomes a striplike end under the appropriate coordinate change. Namely, we view such a neigbourhood (holomorphically) as a neigbourhood of 0 in the upper half plane $\overline{\mathbb{H}}$ (well defined up to rescaling of $\overline{\mathbb{H}}$) which corresponds to a strip by the map $z \mapsto \log z$. Without loss of generality this model is valid on $\{z \in \overline{\mathbb{H}} : |z| \leq 3\}$ and furthermore this region contains only regular values of $\pi$.

Locally near each $z_i$ we take $\widetilde{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ to be the identity away from the $z_i$ and near them to be given by:

$$\begin{aligned} u:\ \widetilde{\mathbb{H}} &\longrightarrow \overline{\mathbb{H}} \\ re^{i\theta} &\mapsto h(r)e^{i\theta} \end{aligned}$$

Here $\widetilde{\mathbb{H}}$ is $\overline{\mathbb{H}} \setminus 0$ with the standard complex structure and $h\colon \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}^{\geq 0}$ is a smooth function such that:

- $h(r) = 0$ for $r \leq 1$ only
- $h(r) = r$ for $r \geq 2$
- $h$ is increasing

We now pullback $(E, \pi)$ to $(\widetilde{E}, \tilde{\pi})$ which is an exact MBL–fibration over a surface with striplike ends.

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\ \widetilde{\mathbb{D}} \arrow[r, "u"] & \overline{\mathbb{D}} \end{tikzcd}$$

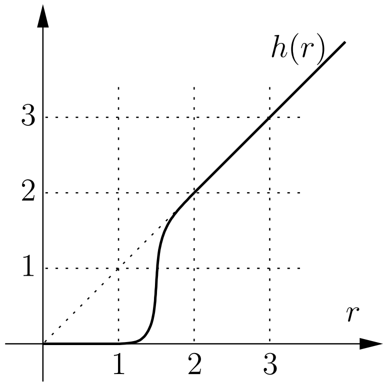


Figure 2: The function $h$.

**Proposition 2.22.** *Consider $(\widetilde{E}, \tilde{\pi})$ as above. Suppose we have a compact connected Lagrangian submanifold in the fibre at infinity over one end of each edge of the base. Then we can construct some $\tilde{\rho}$ and deformation of the exact MBL–fibration, such that:*

- *$L$ extends by symplectic parallel transport to a Lagrangian boundary condition $Q$*
- *$(Q, \tilde{\rho})$ is a well defined enclosed Lagrangian boundary condition.*

*Furthermore the resulting $(\widetilde{E}, \tilde{\pi})$ and $(Q, \tilde{\rho})$ are well-defined up to deformation through such choices.*

It should be noted here that we may, without loss of generality, use Lemma 2.19 to deform $E$ within some finite level set $\rho_{min}$ of $\rho$ to ensure that $Q$ is defined on $\widetilde{E}$ and contained within $u^{-1}\left(\rho^{-1}\left[0, \rho_{min}\right)\right)$. For the rest of the proof of the Proposition we will deform only $u$ and we shall define a $\tilde{\rho}$ with a convex level set contained in $u^{-1}\left(\rho^{-1}\left[\rho_{min}, \rho_{max}\right]\right)$ for any $\rho_{max} > \rho_{min}$.

We will see first how far $u^*\rho$ is from making $Q$ an enclosed Lagrangian boundary condition.

Let $\tilde{J}$ be the complex structure on $\widetilde{E}$ (as given in Lemma 13) and $\tilde{u}^*J$ be the pullback of the complex structure from $E$ where this is defined. Similarly we have the standard complex structure $\tilde{j}$ on $\widetilde{\mathbb{D}}$ and also the pullback $u^*j$ from $\overline{\mathbb{D}}$ where $u$ is an immersion. Let $\tilde{\pi}^*\tilde{j}$, $\tilde{\pi}^*(u^*j)$ be the horizontal lifts $D\tilde{\pi}|_{T\widetilde{E}^h}^{-1} \circ \tilde{j} \circ D\tilde{\pi}$ and $D\tilde{\pi}|_{T\widetilde{E}^h}^{-1} \circ u^*j \circ D\tilde{\pi}$ respectively. It should be noted here that $\tilde{J}$ and $J$ agree on $T\widetilde{E}^v$ and that $\tilde{J} - u^*J = \tilde{\pi}^*\tilde{j} - \tilde{\pi}^*(u^*j)$.

Where $u$ is holomorphic these complex structures agree so $u^*\rho$ is plurisubharmonic. Also where $u$ is locally constant $-d(d(u^*\rho) \circ \tilde{J})$ splits as $u^*\Omega$ on $T\widetilde{E}^v$ and zero horizontally, so $u^*\rho$ is subharmonic. So far so good. The difficulty arises in dealing with the region $\frac{1}{t} \leq r \leq \frac{2}{t}$ in $\widetilde{\mathbb{H}}$. Here we have:

$$
\begin{aligned}
-d(d(u^*\rho) \circ \tilde{J}) &= -d(d(u^*\rho) \circ (\tilde{u}^*J + \tilde{\pi}^*(\tilde{j} - u^*j))) \\
&= \tilde{u}^*\Omega|_{(T\widetilde{E}^v)^{\otimes 2}} + \tilde{u}^*\Omega|_{(T\widetilde{E}^h)^{\otimes 2}} - d(d(u^*\rho) \circ (\tilde{\pi}^*(\tilde{j} - u^*j))
\end{aligned}
$$

Applied to pairs of vectors $(V, \tilde{J}V)$ the first two terms are positive semi-definite (for the second we used that $u$ is nowhere orientation reversing). The third term may not be, but it evaluates to zero on $(T\widetilde{E}^v)^{\otimes 2}$ and we will show how to adjust $-d(d(u^*\rho) \circ \tilde{J})$ by adding a pull back by $\pi$ of a certain functional $\widetilde{\mathbb{D}} \longrightarrow \mathbb{R}$ to $\rho$ to achieve positive semi-definiteness everywhere.

Let $\tilde{R}, \tilde{\Theta}, R, \Theta$ be horizontal lifts of the vector fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ to $\widetilde{E}, E$ respectively. Then at any point $p$ we have:

$$\begin{aligned} \tilde{\pi}^*(\tilde{j} - u^* j): \quad r\tilde{R} &\mapsto \tfrac{h(r) - rh'(r)}{h(r)} \tilde{\Theta} \\ \tilde{\Theta} &\mapsto \tfrac{h(r) - rh'(r)}{rh'(r)} r\tilde{R} \end{aligned}$$

and also:

$$\begin{aligned} d(\tilde{u}^*\rho)|_{T_p\widetilde{E}^h} &= \left(R_{\tilde{u}(p)}(\rho)(\pi^* dr)_{\tilde{u}(p)} + \Theta_{\tilde{u}(p)}(\rho)(\pi^* d\theta)_{\tilde{u}(p)}\right) \circ D\tilde{u}_p|_{T_p\widetilde{E}^h} \\ &= R_{\tilde{u}(p)}(\rho) r h'(r) \left(\tilde{\pi}^* \frac{dr}{r}\right)_p + \Theta_{\tilde{u}(p)}(\rho) \left(\tilde{\pi}^* d\theta\right)_p \end{aligned}$$

Composing the functions defined above yields:

$$\begin{aligned} d(\tilde{u}^*\rho) \circ \tilde{\pi}^*(\tilde{j} - u^* j)_p &= R_{\tilde{u}(p)}(\rho)(h(r) - rh'(r))(\tilde{\pi}^* d\theta)_p \\ &\quad + \Theta_{\tilde{u}(p)}(\rho) \frac{h(r) - rh'(r)}{h(r)} \left(\tilde{\pi}^* \frac{dr}{r}\right)_p \end{aligned}$$

This gives us the following expression for the term of the expansion of $-d(d(u^*\rho) \circ \tilde{J})$ which was potentially not positive-semidefinite (see above).

$$\begin{aligned} -d(d(\tilde{u}^*\rho) \circ \tilde{\pi}^*(\tilde{j} - u^* j))_p &= R_{\tilde{u}(p)}(\rho) r^2 h''(r) \tilde{\pi}^* \left(\frac{dr}{r} \wedge d\theta\right)_p \\ &\quad + R_{\tilde{u}(p)}(R(\rho)) r h'(r)(rh'(r) - h(r)) \tilde{\pi}^* \left(\frac{dr}{r} \wedge d\theta\right)_p \\ &\quad - \Theta_{\tilde{u}(p)}(\Theta(\rho) \frac{rh'(r) - h(r)}{h(r)} \tilde{\pi}^* \left(\frac{dr}{r} \wedge d\theta\right)_p \\ &\quad + d_v(R_{\tilde{u}(p)}(\rho)) \wedge (rh'(r) - h(r)) \tilde{\pi}^* \left(d\theta\right)_p \\ &\quad + d_v(\Theta_{\tilde{u}(p)}(\rho)) \wedge \frac{rh'(r) - h(r)}{h(r)} \tilde{\pi}^* \left(\frac{dr}{r}\right)_p \end{aligned}$$

Here $d_v$ is the differential $d$ evaluated only in the fibre directions.

We do not yet have enough control over these summands, so we consider a one parameter family of maps $u$ converging to the identity map on $\widetilde{H}$. These are defined by replacing the function $h$ with the family of functions

$$h_t(r) = \frac{h(tr)}{t}$$

for $t \in [1, \infty)$.

Considering the dependence on $t$ one now has:

$$\begin{aligned}
-d(d(\tilde{u}^*\rho)\circ\tilde{\pi}^*(\tilde{j}-u^*j))_p &= \frac{1}{t}R_{\tilde{u}(p)}(\rho)(rt)^2h''(rt)\tilde{\pi}^*\left(\frac{dr}{r}\wedge d\theta\right)_p \\
&+\frac{1}{t}R_{\tilde{u}(p)}(R(\rho))(rt)h'(rt)((rt)h'(rt)-h(rt))\tilde{\pi}^*\left(\frac{dr}{r}\wedge d\theta\right)_p \\
&-\Theta_{\tilde{u}(p)}(\Theta(\rho)\frac{(rt)h'(rt)-h(rt)}{h(rt)}\tilde{\pi}^*\left(\frac{dr}{r}\wedge d\theta\right)_p \\
&+\frac{1}{t}d_v(R_{\tilde{u}(p)}(\rho))\wedge((rt)h'(rt)-h(rt))\tilde{\pi}^*(d\theta)_p \\
&+d_v(\Theta_{\tilde{u}(p)}(\rho))\wedge\frac{(rt)h'(rt)-h(rt)}{h(rt)}\tilde{\pi}^*\left(\frac{dr}{r}\right)_p
\end{aligned}$$

With this we can now describe how *small* $-d(d(u^*\rho)\circ\tilde{\pi}^*(\tilde{j}-u^*j))$ is in terms of $t$. To do this we define $||V||$ for $V\in T\widetilde{E}^v$ to be $\tilde{u}^*\Omega(V,\tilde{J}V)$, i.e. the pullback of the metric on fibres of $E$.

**Lemma 2.23.** *Given any $\rho_{max}>\rho_{min}$ there is some constant $K>0$ together with a smooth functional $\alpha\in C^\infty(\widetilde{E})$ and one-forms $\beta,\gamma$ on $\widetilde{E}$ supported over the model neighbourhood $\{z\in\widetilde{\mathbb{H}}:|z|\le 3\}$ such that:*

$$-d(d(\tilde{u}^*\rho)\circ\tilde{\pi}^*(\tilde{j}-u^*j))=\alpha\tilde{\pi}^*\left(\frac{dr}{r}\wedge d\theta\right)+\beta\wedge\tilde{\pi}^*(d\theta)+\gamma\wedge\tilde{\pi}^*\left(\frac{dr}{r}\right)$$

*and*

- *$\beta,\gamma$ evaluate to zero horizontally*
- *$\alpha,\beta,\gamma\equiv 0$ where $r\notin[\frac{1}{t},\frac{2}{t}]$*
- *$|\alpha|\le\frac{K}{t}$ on $\rho^{-1}[0,\rho_{max})$*
- *For any vertical tangent $V\in T\widetilde{E}^v$ we have $|\beta(V)|,|\gamma(V)|\le\frac{K}{2t}||V||$ on $\rho^{-1}[0,\rho_{max})$*

*Proof.* We examine the various summands of

$$-d(d(\tilde{u}^*\rho)\circ\tilde{\pi}^*(\tilde{j}-u^*j))_p$$

as described above. The expressions in terms of $h$ and $rt$ are all smooth as functions of $rt$ and vanish for $1\le rt\le 2$, so must be bounded independently of $t$. By compactness $R_{\tilde{u}(p)}(\rho)$ and $R_{\tilde{u}(p)}(R(\rho))$ have bounds independent of $t$. Similarly for $d_v(R_{\tilde{u}(p)}(\rho))$.

We are interested only in the region $r\in[\frac{1}{t},\frac{2}{t}]$ and $u(r,\theta)=(h_t(r),\theta)$, so we restrict attention to $\tilde{u}^*p$ in fibres where $r\le\frac{2}{t}$.

By compactness $\Theta_{\tilde{u}(p)}(\Theta(\rho))$ and $d_v(\Theta_{\tilde{u}(p)}(\rho))$ are bounded on $\rho^{-1}[0,\rho_{max})$ where $r\le\frac{2}{t}$ and take value 0 in the fibre over 0, so vanish to first order as $r\to 0$. Each summand of $-d(d(\tilde{u}^*\rho)\circ\tilde{\pi}^*(\tilde{j}-u^*j))_p$ is a product of either of these two terms, or $\frac{1}{t}$ with bounded terms, hence the result. □

**Corollary 2.24.** *Let $C>\frac{K^2+K}{t}$, then over the region $\frac{1}{t}\le r\le\frac{2}{t}$ in $\widetilde{\mathbb{H}}$*

$$\omega:=-d(d(\tilde{u}^*\rho)\circ\tilde{J})+\tilde{\pi}^*\left(C\;\frac{dr}{r}\wedge d\theta\right)$$

*gives a positive semi-definite quadratic form on $T\widetilde{E}$ when applied to pairs of vectors $(X,\tilde{J}X)$.*

*Proof.* Split $X = V + H$ into vertical and horizontal components respectively. We will write $||H||$ for the metric $\tilde{\pi}^* \left(\frac{dr}{r} \wedge d\theta\right)(\_, \tilde{J}\_)$ on $T\widetilde{E}^h$.

$$\begin{aligned}
\omega(X, \tilde{J}X) &= \Omega(V, \tilde{J}V) + \Omega(H, \tilde{J}H) \\
&\quad + (C+\alpha)\tilde{\pi}^* \left(\frac{dr}{r} \wedge d\theta\right)(H, \tilde{J}H) \\
&\quad + \beta \wedge \tilde{\pi}^* (d\theta)(V, \tilde{J}H) + \beta \wedge \tilde{\pi}^* (d\theta)(H, \tilde{J}V) \\
&\quad + \gamma \wedge \tilde{\pi}^* \left(\frac{dr}{r}\right)(V, \tilde{J}H) + \gamma \wedge \tilde{\pi}^* \left(\frac{dr}{r}\right)(H, \tilde{J}V) \\
&\geq ||V||^2 - \frac{2K}{t} ||V|| \, ||H|| + (C - \frac{K}{t}) \, ||H||^2
\end{aligned}$$

Examination of the discriminant shows this is $\geq 0$ for all $X$ if $C > \frac{K^2+K}{t} > \frac{K^2}{t^2} + \frac{K}{t}$. □

We are now ready to define $\tilde{\rho}$ for large enough $t$ and proceed with the proof of Proposition 2.22.

*Proof.* We will construct a family of functions $\tilde{g}_t \in C^\infty(\widetilde{\mathbb{H}})$ and define $\tilde{\rho} := \tilde{g}_t \circ \tilde{\pi} + \tilde{u}^* \rho$. For large enough values of $t$ this will have the necessary properties to achieve convexity on a level set of $\tilde{\rho}$ contained in $\tilde{u}^{-1}\rho^{-1}[\rho_{min}, \rho_{max}]$.

It is now convenient to change coordinates by the exponential map

$$\widetilde{\mathbb{H}} \longleftarrow \{z \in \mathbb{C} : \operatorname{im} z \in [0, \pi]\}$$

Using coordinates $z = x + iy$ on the strip, $r = 1, 2, 3$ corresponds to $x = 0, \log 2, \log 3$. We define $g$ to be a scalar function on the strip, such that for some positive $\epsilon < \frac{\log 3 - \log 2}{2}$:

(i) $g$ depends only on $x$ and is increasing in $x$

(ii) $g(x, y) = 0$ if $x \geq \log 3$

(iii) $g(x, y) = \rho_{min} - \rho_{max}$ if $x \leq -\epsilon$

(iv) $\exists \delta > 0$ with $\frac{\partial^2}{\partial x^2} g \geq \delta$ for $x \in [0, \log 2]$

(v) $\frac{\partial^2}{\partial x^2} g = 0$ for $x \in [\log 2 + \epsilon, \log 3 - \epsilon]$

(vi) $\frac{\partial^2}{\partial x^2} g$ is non negative away from $x \in [\log 3 - \epsilon, \log 3]$

These conditions are illustrated in Figure 3. Such a function is easily constructed

Now we extend $g$ to $g_T$, a one-parameter family of functions parametrised by $T \in [0, \infty]$. Let $L := \max\left(-\frac{\partial^2}{\partial x^2} g\right)$ and

$$F := 1 + \frac{(g(\log 3 + \epsilon) - g(\log 2 + \epsilon))\left(\frac{T}{\log 3 - \log 2 - 2\epsilon} + 2\right)}{\rho_{max} - \rho_{min}}$$

Then we can define smooth $g_T$ by:

- $g_T(x, y) = \frac{g(x+T, y) + \rho_{max} - \rho_{min}}{F} - \rho_{max} + \rho_{min}$ for $x \leq \log 2 + \epsilon - T$

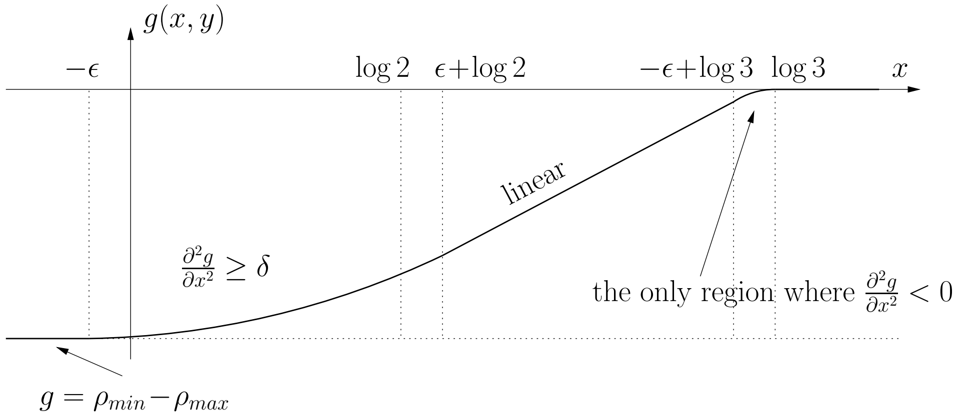


Figure 3: The function $g$ in terms of $x$.

- $g_T(x,y) = \frac{g(x+T,y)}{F}$ for $x \geq \log 3 - \epsilon$
- $g_T$ interpolates linearly in the range $x \in [\log 2 + \epsilon - T, \log 3 - \epsilon]$

For some rather messy constant $S > 0$ and large enough $T$:

- $\frac{\partial^2}{\partial x^2} g_T$ is non negative away from $x \in [\log 3 - \epsilon, \log 3]$
- $\frac{\partial^2}{\partial x^2} g_T \geq \frac{-L}{F} \geq -\frac{SL}{T}$
- $\frac{\partial^2}{\partial x^2} g_T > \frac{\delta}{F} \geq \frac{S\delta}{T}$ for $x \in [-T, \log 2 - T]$

Let $\tilde{g}_T$ be $g_T$ pushed forward to $\widetilde{\mathbb{H}}$ by the exponential map, then we have:

$$-d(d\tilde{g}_T \circ \tilde{j})_{(r,\theta)} = \left( \frac{\partial^2}{\partial x^2} g_T(\log r, \theta) \right) \frac{dr}{r} \wedge \theta$$

Setting $T = \log t$ and we now consider the function $\tilde{u}^*\rho + \tilde{\pi}^*\tilde{g}_{\log t}$ on the set $\tilde{u}^{-1}\rho^{-1}[\rho_{min}, \rho_{max}]$. We see that for large enough $t$ we have $\frac{S\delta}{\log t} > \frac{K^2+K}{t}$, so by Corollary 2.24 it is plurisubharmonic over the region $r \in \left[\frac{1}{t}, \frac{2}{t}\right]$. Also if $t$ is small enough, then by compactness it is plurisubharmonic over the region $r \in \left[\frac{2}{t}, \frac{3}{t}\right]$. Elsewhere it is subharmonic.

The $\rho_{max}$ level set of $\tilde{u}^*\rho + \tilde{\pi}^*\tilde{g}_{\log t}$ is contained in $\tilde{u}^{-1}\rho^{-1}[\rho_{min}, \rho_{max}]$. It agrees with the $\rho_{min}$ level set of $\tilde{u}^*\rho$ near $r = 0$ and the $\rho_{min}$ level set away from the neighbourhood modelled by $\widetilde{\mathbb{H}}$. Hence it makes our Lagrangian boundary condition enclosed.

A careful examination of this procedure shows that the choices involved are all canonical up to isotopy through such choices. □

**Remark 2.25.** As $t \longrightarrow \infty$ the above argument gives a deformation of MBL–fibrations from the fibration with striplike ends to the original one over $\overline{\mathbb{D}}$ with marked points on the boundary. For each value of $t$ we also have the same Lagrangian boundary condition. With a little care, we also get a deformation of *enclosed* Lagrangian boundary conditions as $t \longrightarrow \infty$. This will be important later as it shows the relative invariants from these fibrations are all the same.

Now we have shown how to deform our original model fibration over $\overline{\mathbb{D}}$ to have striplike ends, the trivialisations over which are specified entirely by the image in $N$ of the end and the fibre of $E \longrightarrow N$ over that point. Suppose we now have a pair of holomorphic maps $\overline{\mathbb{D}} \longrightarrow N$

which agree on one point $z \in \partial\mathbb{D}$, then by deforming these as above to admissible maps from surfaces with striplike ends we can glue at the ends corresponding to $z$. Furthermore, using the previous Lemmas we can still define enclosed Lagrangian boundary conditions on the resulting fibration simply by specifying some Lagrangians in the fibres at infinity. The same works for larger composites.

In order to define an exact MBL–fibration from any admissible map $u\colon B \longrightarrow N$ we work with composites of our model maps (as illustrated in Figure 4). Namely, any admissible $u$ is isotopic through admissible maps to a map obtained as follows. Take an acyclic collection of holomorphically embedded discs in $N$ joined (without any condition on tangencies) at certain marked points on their boundaries. By this, I mean that the graph whose vertices are given by the discs and whose edges correspond to marked points at which the discs are joined, is acyclic. We shall call this a *tree construction*. Now:

- pullback the fibration $E \longrightarrow N$ over each of these discs
- deform them all to have striplike ends instead of each marked point
- glue the fibrations at the striplike ends corresponding to the joined marked points

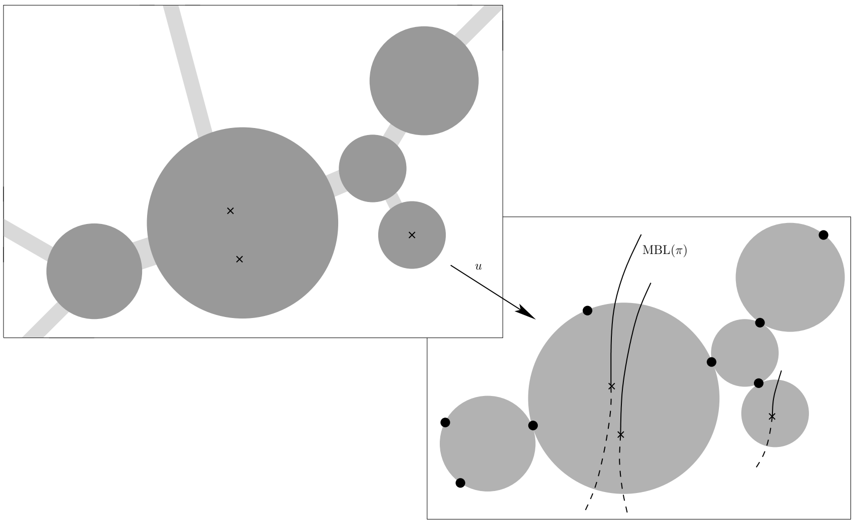


Figure 4: An example of a map of a surface $B$ into $N$ given by a tree construction. Trivialisations of the exact MBL–fibration are over lighter shaded regions.

Let $u\colon B \longrightarrow N$ be such a construction. This specifies for each striplike end $e \in I$ a regular point $z_e \in N$ to which it maps and an exact MBL–fibration $(E, \pi)$ over $B$. The choices involved in the construction mean that $(E, \pi)$ is well defined up to isotopy. The same is true if we choose sufficiently many exact Lagrangian submanifolds in the fibres over the points $z_e$ and a suitable deformation (by Lemma 2.19) to define an enclosed exact Lagrangian boundary condition $Q$. One can always enlarge the support of this deformation such that $Q$ remains defined through the relevant isotopies of exact MBL–fibrations (and all subsequent arguments in this section).

Much of the tree structure and the positioning of the singular values of $\pi$ in $B$ by the construction does not affect the exact MBL–fibration except by isotopy. Suppose we have a smooth embedding of $f\colon B \longrightarrow B$ with following properties (e.g. Figure 5):

(i) $u \circ f$ should be admissible and define the same set $\{z_e\}$ of ends. In particular $f(\partial B)$ should not contain any critical value of $\pi$.

(ii) Consider any of the regions of $B$ identified with parts of $\widetilde{\mathbb{H}}$ in order to construct the striplike ends (including those glued together). In each of these regions $f(B)$ should be a union of disjoint wedges. This corresponds in the trivialisation of the striplike ends to $f(B)$ being a union of 'substrips'.

(iii) $f$ should be isotopic to the identity through smooth embeddings maintaining condition (i) above.

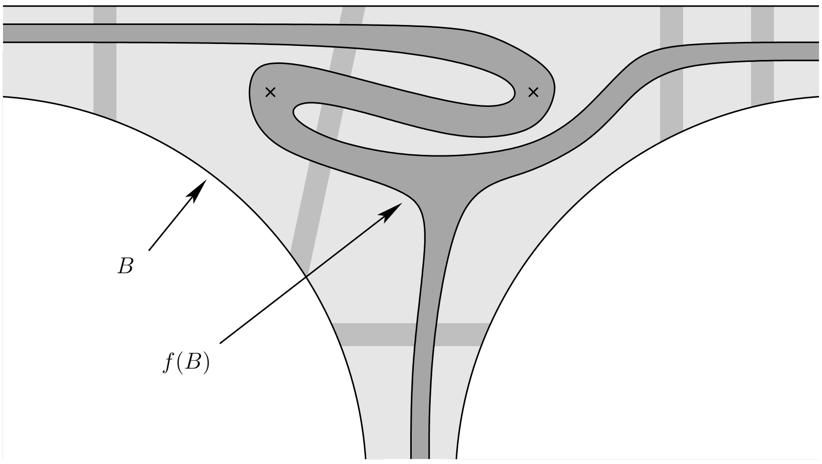


Figure 5: An example restriction of the base of a tree construction used to change the tree structure (it redistributes the singular values). The regions where striplike ends were glued in the construction of $B$ are indicated by shading

**Lemma 2.26.** *In the above construction, the isotopy of (iii) induces an isotopy of exact MBL–fibrations (and also of any enclosed exact Lagrangian boundary conditions, provided sufficient care is taken with symplectic parallel transport using Lemma 2.19).*

The resulting fibration can be viewed as another potentially very different tree construction.

*Proof.* This proof of this is immediate from the definitions above. To construct the accompanying isotopy of enclosed exact Lagrangian boundary conditions, one simply fixes an exact Lagrangian in the fibre over one end of each edge of the base and extends by symplectic parallel transport, at each stage of the isotopy, to define $Q$. This works with a sufficiently strong application of Lemma 2.19. ☐

**Lemma 2.27.** *Suppose $\tilde{u}\colon \tilde{B} \longrightarrow N$ is another such construction such that $u, \tilde{u}$ are isotopic through admissible maps fixing the endpoints $z_e$. Then $u, \tilde{u}$ are related by a finite sequence of the following 'moves':*

*(a) isotopy of the tree construction of embedded admissible holomorphic discs through such constructions*

*(b) decomposition of any of the embedded holomorphic discs into two joined at a new marked point (see Figure 6)*

*(c) changing the tree structure by restriction of the base to a surface embedded in it as in Lemma 2.26*

*Suppose furthermore we have enclosed exact Lagrangian boundary conditions $Q, \tilde{Q}$ defined on the exact MBL–fibrations over $u, \tilde{u}$ by the same set Lagrangians in the fibres of $N$ over endpoints $z_e$ which are not 'internal' to the tree constructions. Then the two exact MBL–fibrations, together with enclosed exact Lagrangian boundary conditions, are isotopic.*

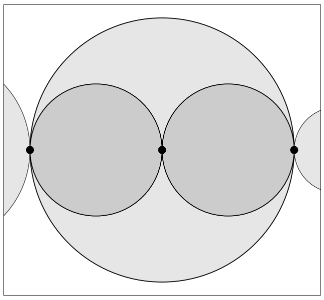

Figure 6: An example decomposition of a holomorphic disc with boundary marked points into two joined at a new marked point

*sketch of proof.* We take the base of one of the fibrations and smoothly embed it in the base of the other as for Lemma 2.26. This can be done such that after application of move (c) the tree constructions are now related only by the remaining two moves. It may help to think of decomposing and isotoping the tree structures such that all the discs are complex linear embeddings in our favourite coordinate neighbourhoods of $N$. $\square$

## 2.3 Holomorphic sections

In this section I define the relative invariant associated to an exact MBL–fibration with enclosed exact Lagrangian boundary conditions. The invariant's definition will be largely identical to Seidel's construction of relative invariants of exact Lefschetz fibrations (cf. [2] sections 2.1 and 2.4). I begin by defining a large class of almost complex structures on an exact MBL–fibration $(E, \pi)$ which extend $J_0$.

**Definition 2.28.** An *almost complex structure* on an exact MBL-fibration $(E, \pi)$ over a surface $B$ with striplike ends and boundary marked points $P \subset \partial B$ is an almost complex structure $J$ on $\pi^{-1}(B \setminus P)$ such that:

- $J = J_0$ in a neighbourhood of $\mathrm{Crit}(\pi)$ and on the complement of some compact subset of $E$
- $D\pi \circ J = j \circ D\pi$, i.e $\pi$ is $(J, j)$–holomorphic
- Over striplike ends $J$ is invariant w.r.t. translation by $\mathbb{R}^{\pm}$
- Over embedded curves $\gamma$ in $B$ ending, with non-zero derivative, at a boundary marked point $e$, $J$ extends smoothly to an almost complex structure $J_{e,t}$ in the fibre over $e$ depending only on the angle of approach $\pi t \in [0, \pi]$.

The last two conditions ensure that, at each end, $J$ limits to some time dependent almost complex structure $J_t$ in the fibre over the end.

**Definition 2.29.** An almost complex structure $J$ on $E$ is *compatible relative to* $j$ if the 2–form $\Omega(.,J.)|_{(TE^v)^{\otimes 2}}$ is everywhere symmetric and positive definite. We will denote the set of such almost complex structures by $\mathcal{J}(E,\pi)$.

Given a *time-dependent* almost complex structure $J_{e,t}$ in the fibre at infinity or fibre over the marked point at each end $e$ of $B$, the set of those $J$ which agree in the limit with each $J_{e,t}$ will be denoted $\mathcal{J}(E,\pi,\{J_{e,t}\}_{e\in I})$ or simply $\mathcal{J}(E,\pi,\{J_e\})$.

Let $M$ be a Kähler manifold and $L_1, L_0$ a transversely intersecting pair of compact connected Lagrangian submanifolds satisfying the conditions of Remark 2.1. The differential in Floer cohomology $HF^*(L_1,L_0)$ is defined using the moduli spaces $\mathcal{M}_J(x,y)$ of strips $u\colon \mathbb{R}\times[0,1]\longrightarrow M$ with edges on $L_1$ and $L_0$ which interpolate between intersections $x$ and $y$ and satisfy a Cauchy-Riemann equation

$$\frac{\partial}{\partial s}u + J_t(u)\frac{\partial}{\partial t}u = 0$$

Here $J=(J_t)_{t\in[0,1]}$ is a generic 1–parameter family of compatible almost complex structures on $M$.

One obtains the identical moduli spaces by counting holomorphic sections of the trivial fibration $M\times\mathbb{R}\times[0,1]\longrightarrow\mathbb{R}\times[0,1]$. To get this equivalence one takes the standard complex structure $j$ on the base and at points in the fibre over $(s,t)\in\mathbb{R}\times[0,1]$ an almost complex structure which splits as $j$ horizontally and $J_t$ vertically. For Lagrangian boundary condition one takes $L_1\times\{1\}\times\mathbb{R}$ and $L_0\times\{0\}\times\mathbb{R}$. Holomorphic convexity of $M$ implies enclosedness of the Lagrangian boundary condition.

**Remark 2.30.** This complex structure on a trivial fibration over the infinite strip is $\mathbb{R}$–equivariant where $\mathbb{R}$ acts by translation in the $s$–coordinate. It is this equivariance and the triviality which makes it the correct model for our almost complex structures over striplike ends since it causes degenerations of sections to be pairs consisting of a section and an element of some $\mathcal{M}_J(x,y)$ (up to translation).

I shall generally refer to Floer cohomology with differential calculated by counting $\mathbb{R}$ orbits of holomorphic sections of a trivial fibration over $\mathbb{R}\times[0,1]$. Complex structures used in Floer cohomology will be those complex structures on the fibration which correspond to one parameter families $(J_t)_{t\in[0,1]}$ in the usual setting.

For a compatible almost complex structure $J$ on an exact MBL–fibration $E$, we denote the set of smooth sections of $E$, which are $(j,J)$–holomorphic away from boundary marked points, by $\mathcal{M}_J$. We refer to these simply as *holomorphic sections*. We denote the space of those holomorphic sections with boundary on a particular Lagrangian boundary condition by $\mathcal{M}_J(Q)$. This has the $C^\infty$–topology. If $Q$ is enclosed we will write $\mathcal{M}_J(Q)$ for the subspace of those sections contained within the particular enclosure. It will be clear from the context which enclosure is being used.

**Lemma 2.31.** *(cf. [2] Lemma 2.2)*

*Let $(E,\pi)$ be an exact MBL–fibration over a surface with ends, exact Lagrangian boundary conditions $(Q,\rho,R)$, and almost complex structures $J_{e,t}$ specified on the striplike ends $e\in I$. There is $\epsilon>0$ such that for every $J\in\mathcal{J}(E,\pi,\{J_{e,t}\}_{e\in I})$ the space of $(j,J)$–holomorphic sections $u$ with the given Lagrangian boundary condition decomposes into two components:*

- *those with image contained in the fibrewise compact set* $\rho^{-1}[0, R-\epsilon]$
- *those whose image leaves* $\rho^{-1}[0, R]$

*Proof.* Subharmonicity of $\rho$ w.r.t. $J$ implies that $\rho \circ u$ is subharmonic so cannot have a local maximum in the region $\rho^{-1}[R-\epsilon, R]$. □

In the construction of Floer cohomology it is necessary to use a *regular* almost complex structure. This ensures that the moduli spaces of sections are smooth finite dimensional manifolds and, in particular, that the Floer differential squares to zero. Here we have the same requirement.

We say that a compatible almost complex structure $J$ on an exact MBL–fibration is regular at $u \in \mathcal{M}_J$ if $D_{u,J}$ is onto. Here $D_{u,J}$ is the extension of the linearised $\bar{\partial}_J$ operator at the $J$–holomorphic section $u$ to relevant Sobolev completions. For a more detailed description of regularity which applies in this setting, see [2, Section 2]. This condition ensures that $\mathcal{M}_J$ is a smooth finite dimensional manifold of the *correct* dimension (given by a Maslov index of $u$) near $u \in \mathcal{M}_J$. $J$ is regular with respect to enclosed Lagrangian boundary condition $Q$ if it is regular for all $u \in \mathcal{M}_J(Q)$. We denote the space of such almost complex structures by

$$\mathcal{J}^{reg}(E, \pi, Q, \{J_{e,t}\}) \subset \mathcal{J}(E, \pi, \{J_{e,t}\})$$

Corresponding to Lemma 2.20 of [2] and with essentially the same proof, we have generic regularity for complex structures in $\mathcal{J}(E, \pi, \{J_e\})$ given any transverse enclosed exact Lagrangian boundary condition $(Q, \rho, R)$.

**Lemma 2.32.** $\mathcal{J}^{reg}(E, \pi, Q, \{J_e\})$ *is* $C^\infty$*–dense in* $\mathcal{J}(E, \pi, \{J_e\})$*. More precisely, given some non-empty open subset* $U \subset B$ *which is disjoint from the ends and a* $J \in \mathcal{J}(E, \pi, \{J_e\})$ *there are* $J' \in \mathcal{J}^{reg}(E, \pi, Q, \{J_e\})$ *arbitrarily* $C^\infty$*-close such that* $J = J'$ *outside* $\pi^{-1}(U)$*.*

By bounding the symplectic action, as in [2] one also achieves a compactification of $\mathcal{M}_J(Q)$ in the *Gromov-Floer topology* by adding broken sections.

Counting isolated holomorphic sections in $\mathcal{M}_J(Q)$ which limit to given sets $\{x_e\}_{e \in I}$ of intersection points in the fibres at the ends (denote these subsets $\mathcal{M}_J(Q, \{x_e\})$) one defines a linear map on the level of Floer cochain complexes. In fact, by a standard argument, considering degenerations of 1–dimensional families of sections, one shows that it is a chain map.

$$\bigotimes_{e \in I^-} CF(Q_e^1, Q_e^0, J_{e,t}) \xrightarrow{C\Phi_0^{rel}((E,\pi),(Q,\rho,R),J)} \bigotimes_{e \in I^+} CF(Q_e^1, Q_e^0, J_{e,t})$$

$$\otimes_{e \in I^-} \langle x_e \rangle \longmapsto \sum_{\{y_e\}_{e \in I^+}} \#\mathcal{M}_J(Q, \{x_e\} \cup \{y_e\})(\otimes_{e \in I^+} \langle y_e \rangle)$$

The above is written out in full detail, since, in particular, the differentials in the Floer cochain complexes $CF(L, L', J_t)$ depend on the choice of almost complex structure. When one changes this, an argument using continuation maps gives a chain homotopy equivalence to the new cochain complex which is canonical up to chain homotopy. Similarly, compactly supported Hamiltonian symplectomorphisms of either $L$ or $L'$ induce canonical chain homotopy classes of chain homotopy equivalences. Lemma 2.38 (below) can be viewed as a generalisation of both of these results in the setting of exact Lagrangian submanifolds of exact symplectic manifolds.

**Definition 2.33.** The *relative invariant* $\Phi_0^{rel}((E,\pi),(Q,\rho,R),J)$ is defined to be the map induced on Floer cohomology by $C\Phi_0^{rel}((E,\pi),(Q,\rho,R),J)$. Under the assumption that the $Q$ is spin in any fibre, one can also orient the moduli spaces and perform the count of sections with signs (see [14]). This allows the use of $\mathbb{Z}$ coefficients. Alternatively one must restrict to characteristic 2.

**Remark 2.34.** The convention of signs and arrangement of $Q_e^0, Q_e^1$ differs by a 180° rotation from that in [2].

**Remark 2.35.** There is a Poincaré duality arising from trivial fibrations over an infinite strip with both ends in $I^-$ (or both in $I^+$). Composition with this (see gluing below) allows us to swap ends back and forth between $I^+$ and $I^-$ or also to view the relative invariant as the element induced in cohomology by:

$$C\Phi_0^{rel} \in \bigotimes_{e\in I} CF(Q_e^1, Q_e^0)$$

Composing two exact MBL–fibrations with enclosed Lagrangian boundary conditions can be done by gluing over a single striplike end (if necessary one uses Poincaré duality to move other ends out of the way). Corresponding to Proposition 2.2 of [2] we have:

**Lemma 2.36.** *Gluing two exact MBL–fibrations with enclosed Lagrangian boundary conditions together along an oppositely oriented, but otherwise identical striplike end gives the composition of the* $\Phi_0^{rel}((E,\pi),(Q,\rho,R),J)$ *maps.*

*(It is important that all the data of* $E,\pi,Q,J\ldots$*etc agrees where the gluing occurs.)*

We shall now show that a variety of changes can be made to $((E,\pi),(Q,\rho,R),J)$ without changing the relative invariant (beyond composing on either side with the canonical isomorphisms on Floer cohomology).

**Lemma 2.37.** $C\Phi_0^{rel}((E,\pi),(Q,\rho,R),J)$ *is unchanged when one switches the enclosure* $(Q,\rho,R)$ *for an equivalent enclosure.*

*Proof.* Definition 2.13 ensures that the moduli spaces $\mathcal{M}_J(Q,\{x_e\})$ are unaffected by this. ☐

**Lemma 2.38.** *The map* $\Phi_0^{rel}((E,\pi),(Q,\rho,R),J)$ *depends only on the enclosed region* $\rho^{-1}[0,R]$*. Furthermore it is independent (up to composition on either side with the canonical isomorphisms on Floer cohomology) of isotopy of the combined data* $((E,\pi),(Q,\rho,R),J)$ *such that:*

*(a) the data remains valid at all stages for the definition of* $\Phi_0^{rel}((E,\pi),(Q,\rho,R),J)$

*(b) the induced isotopies at each end* $E_e$ *fix the symplectic form and vary the Lagrangian submanifolds only by compactly supported Hamiltonian isotopy*

*Proof.* When one fixes the data over the ends, a standard argument counting sections at all stages of (some perturbation) of this isotopy gives a homotopy of the relative invariant at the level of chain complexes. More generally, it gives a homotopy composed with continuation maps (yielding the canonical isomorphisms) on Floer cohomology groups over the ends. ☐

**Remark 2.39.** In fact, the above argument works to describe general deformations of the data $(E,\pi)$, $(Q,\rho,R)$ and $J$. Suppose for example, we do not require the symplectic forms on fibres over the ends to remain fixed (then we most likely also have to vary $Q$ over the ends to ensure that it remains a Lagrangian boundary condition). The relative invariant then varies by left-

and right-composition with the continuation maps from these changes. In some cases (such as deforming $\Omega$ through exact Kähler forms with the added condition that components of $Q$ have vanishing first cohomology over $\mathbb{R}$) these continuation maps are still canonical isomorphisms, but in general one cannot expect that to be the case.

Suppose the geometric data defining the exact MBL–fibration and boundary conditions splits as a some sort of product, then one can often correspondingly split the relative invariant. A simple example of this is demonstrated in the following Lemma and a more involved version comes in Section 3.

**Lemma 2.40.** *Suppose:*

- $(E, \pi)$ *splits as a smooth fibre product of MBL–fibrations* $(E_1, \pi_1)$ *and* $(E_2, \pi_2)$ *over the same base* $(B, j)$
- *the exact Lagrangian boundary condition* $Q$ *splits as a fibre-product of exact Lagrangian boundary conditions* $Q_1, Q_2$ *in the two factors*
- $(\rho, R), (\rho_1, R), (\rho_2, R)$ *are enclosures for* $Q, Q_1, Q_2$ *respectively with the property that* $\rho$ *is* $C_0$*–close to* $\max\{\rho_1, \rho_2\}$ *in some neighbourhood of* $\rho^{-1}(R)$*.*

*Then the relative invariant* $C\Phi_0^{rel}((E, \pi), (Q, \rho, R), J)$ *splits as the product of the relative invariants on the two factors*

$$C\Phi_0^{rel}((E_1, \pi_1), (Q_1, \rho_1, R), J) \otimes C\Phi_0^{rel}((E_2, \pi_2), (Q_2, \rho_2, R), J)$$

*Proof.* This is a simple generalisation of the corresponding product formula for Floer cohomology (which one has over the ends in the setting of the Lemma). $(E, \pi)$ is smooth whenever $(E_1, \pi_1)$ and $(E_2, \pi_2)$ share no singular values in the base.

Let $J_1 \in \mathcal{J}^{reg}(E_1, \pi_1, Q_1)$, $J_2 \in \mathcal{J}^{reg}(E_2, \pi_2, Q_2)$ be any regular almost complex structures on $E_1$, $E_2$ for boundary conditions $Q_1$, $Q_2$ respectively. We define the almost complex structure $J$ for the fibre product $(E, \pi)$ by pullback from the embedding $E = E_1 \underset{B}{\times} E_2 \longrightarrow E_1 \times E_2$. Compatibility of $J_1$ and $J_2$ relative to $j$ implies that $J$ on the product restricts well to $E$ and is compatible relative to $j$.

The $J$–holomorphic sections of $E$ are in bijection with the pairs of holomorphic sections of $E_1$ and $E_2$. The regularity of $J$ for Lagrangian boundary condition $Q_1 \underset{\partial B}{\times} Q_2$ is an immediate consequence of the regularity of $J_1$ and $J_2$ since the linearisation at any holomorphic section splits as that of $J_1$ and $J_2$, both of which (when extended to the relevant Sobolev spaces) are surjective. This bijection of moduli spaces identifies the deformation theories of the sections. □

**Remark 2.41.** In the case where $\rho_1, \rho_2$ are plurisubharmonic near their respective $R$-level sets and the complex structures near there are integrable, there is always a plurisubharmonic $C_0$–approximation to $\max\{\rho_1, \rho_2\}$ near its $R$-level set (see [15, Lemma 3.8]).

**Lemma 2.42.** *Trivial MBL–fibrations over the infinite strip (with trivial boundary conditions) induce the identity map on Floer cohomology.*

*Proof.* One may take the complex structure for the fibration from that used in calculating the Floer Homology at the striplike ends. The only isolated holomorphic sections (since we do not quotient by the $\mathbb{R}$–action on the base) are the constant sections. These give us the identity map. □

## 2.4 Equivariant and horizontal regularity

In [2] calculations of the relative invariant are made easier by using almost complex structures which complexify the symplectic connection. In [16] $\mathbb{Z}/2$–equivariant almost complex structures are used to calculate Floer cohomology. This section shows how the two ideas work naturally together in the context of the relative invariant.

**Definition 2.43.** $J \in \mathcal{J}(E, \pi)$ is *horizontal* if it preserves $TE^h$. In particular this gives a complex structure to $TE^h$. We denote the space of horizontal almost complex structures which are compatible relative to $j$ by $\mathcal{J}^h(E, \pi)$.

Let $\mathcal{M}^h$ be the set of horizontal sections of $E$, i.e. smooth sections which are integral surfaces for $TE^h$. These are $J$–holomorphic for any horizontal $J$. Similarly let $\mathcal{M}^h(Q)$ be the set of those horizontal sections with boundary in $Q$.

The horizontal sections are easy to work with. In particular there can be at most one horizontal section through each point of a given fibre.

The condition for horizontality is satisfied pointwise by a contractible, non-empty set of possible almost complex structures. Therefore, horizontal almost complex structures exist globally and form a non-empty contractible subspace of $\mathcal{J}(E, \pi)$.

Lemma 2.26 of [2] shows that generic horizontal almost complex structures are regular at all non-horizontal sections of exact Lefschetz fibrations. Furthermore, Proposition 2.28 of [2] details conditions under which one also has regularity at the horizontal sections. Rather than just repeat this for MBL–fibrations, we will also detail conditions under which these regularity results can be achieved using equivariant almost complex structures for some group action.

**Definition 2.44.** We say a group $G$ acts on an exact MBL-fibration $(E, \pi)$ if it acts smoothly on $E$ preserving all the structure $(\Omega, \Theta, J_0)$ and induces a well-defined action on $B$ by $j$-biholomorphic maps.

We also require that the action induces a well-defined action of $G$ on the fibre over each end. In the trivialised regions over any striplike ends, we require further that the action split as an action on the fibre and locally an action by translation in the $\mathbb{R}^{\pm}$ direction along the end.

An enclosed Lagrangian boundary condition $(Q, \rho, R)$ is $G$–equivariant if $Q$ and $\rho$ are.

This definition allows for both fibrewise actions and reparametrisations of the base of the exact MBL–fibration. The group of reparametrisations of the base will generally be trivial or $\mathbb{R}$ acting by translation. It is nonetheless useful to phrase things such that the same technique applies in either case.

$G$ also acts on the space of sections of $E$. When $J$ is $G$–equivariant this action restricts to $\mathcal{M}_J(Q)$. Denote the set of such $J$ by $\mathcal{J}^G(E, \pi, (J_e)_{e \in I})$. For example, in defining the differential on Floer cohomology, one has $G = \mathbb{R}$ acting by translations of the infinite strip. The differential is given by counting isolated $\mathbb{R}$–orbits in $\mathcal{M}_J(Q)$.

In general, one can use this action to make statements about Floer cohomology and/or the relative invariant as long as it is possible to find an almost complex structure that is both $G$–equivariant and regular. Proposition 2.45 below combines ideas from of [16] and [2] to give an example of the sort of situation in which generic $J \in \mathcal{J}^{h,G}(E, \pi, (J_e)_{e \in I})$ are regular at non-horizontal $u \in \mathcal{M}_J(Q)$. Separate conditions for regularity at all $u \in \mathcal{M}^h(Q)$ are also given.

Given a section $u$ we define the set of regular points $\mathrm{Reg}(u)$ of $u$ to be the subset of points $z \in B$ where

- $Du(T_z B)$ is not horizontal (i.e. not a subset of $T_{u(z)} E^h$,

- for all $g \in G$, $w \in B$ we have $u(z) = g(u(w))$ if and only if $z = w$ and $g$ is the identity,
- there exists an open ball $U \subset E$ containing $u(z)$ such that the set of $(g, w) \in G \times B$ with $g(u(w)) \in \overline{U}$ is compact.

The last condition can be thought of as an extension of the previous condition, designed to stop $u(z) = g(u(w))$ "occurring at infinity in $G \times W$".

**Proposition 2.45.** *Suppose a group $G$ acts on an exact MBL–fibration $(E, \pi)$ with $G$–equivariant transverse enclosed Lagrangian boundary condition $(Q, \rho, R)$. Let $U \subset B$ be a connected open $G$–invariant subset of the base. Then, for $J \in \mathcal{J}^{h,G}(E, \pi)$, there are $J' \in \mathcal{J}^{h,G}(E, \pi)$ arbitrarily $C^\infty$–close to $J$, such that $J = J'$ outside $\pi^{-1}(U)$, which are regular at all $u \in \mathcal{M}_J(Q) \setminus \mathcal{M}^h(Q)$ provided that the following conditions hold:*

- *any partial section $w\colon U \longrightarrow E|_U$ which is horizontal and respects the boundary condition (i.e. $w(\partial B \cap U) \subset Q$) is the restriction of some $u \in \mathcal{M}^h(Q)$,*
- *for $u \in \mathcal{M}_J(Q) \setminus \mathcal{M}^h(Q)$ the set $\mathrm{Reg}(u)$ is open and dense in $B$.*

*Suppose also that the symplectic curvature is identically zero in a neighbourhood of the image of any horizontal section $u \in \mathcal{M}^h(Q)$. Then $J'$ is regular at all $u \in \mathcal{M}^h(Q)$.*

*Proof.* By Lemma 2.31 and compactness of the moduli spaces (cf. Lemma 2.3 of [2]), there is an open subset $V \subset E$ disjoint from $\mathrm{Crit}(\pi)$ and contained inside the $R$ sublevel set of $\rho$, such that $\mathrm{im}(u) \subset V$ for each $u \in \mathcal{M}_J$. A modified version of the compactness argument shows this remains true for all almost complex structures in $\mathcal{J}^{h,G}(E, \pi, (J_e)_{e \in I})$ which are sufficiently $C^\infty$–close to $J$. We shall make $J$ regular by perturbing it in $V \cap \pi^{-1}(U)$.

Let $\mathcal{T}_J$ be the tangent space of $\mathcal{J}(E, \pi, j, Q)$ at some $J$ and $\mathcal{T}_J^{h,G}$ the tangent space to $\mathcal{J}^{h,G}(E, \pi, j, Q)$. Horizontality allows us to express $\mathcal{T}_J^{h,S^1}$ as a subspace of $C^\infty(\mathrm{End}(TE^v))$.

For any horizontal, $G$–equivariant $J$ and any holomorphic section $u \in \mathcal{M}_J$, one defines an operator

$$D^{univ}_{u,J}\colon \mathcal{W}^1_u \times \mathcal{T}_J \longrightarrow \mathcal{W}^0_u$$

$$D^{univ}_{u,J}(X, Y) = D_{u,J}(X) + \frac{1}{2} Y \circ Du \circ j$$

The main point of the proof of regularity (cf. [2] and [17]) is to show that $D^{univ}_{u,J}$ always surjects. In our case we wish to show that the restriction

$$\widetilde{D}^{univ}_{u,J}\colon \mathcal{W}^1_u \times \mathcal{T}^{h,G}_J \longrightarrow \mathcal{W}^0_u$$

always surjects. We will prove this by contradiction. Namely, let $F \longrightarrow B$ be the bundle dual to $\Lambda^{0,1}(u^* TE^v)$, then $(\mathcal{W}^0_{u,J})^* = L^q(F)$ (for appropriate $q$). Assuming the map does not surject for some $u, J$ (where $u$ is non-horizontal), there must exist non-zero $\eta \in L^q(F)$ which is orthogonal to the image, i.e.

$$\left(\widetilde{D}^{univ}_{u,J}\right)^* (\eta) = 0 \ \text{ on } B \setminus \partial B \quad \text{and} \quad \int_B \langle \eta, Y \circ Du \circ j \rangle = 0 \ \text{ for all } Y \in \mathcal{T}^{h,G}_J$$

Furthermore, $\eta$ satisfies a $\bar{\partial}$–equation which gives it the unique continuation property. The first equation implies $\eta$ is smooth away from $\partial B$.

Suppose we have a regular point $z \in \mathrm{Reg}(u) \cap (U \setminus \partial B)$ in the support of $\eta$. Then we have $Du(T_z B) \not\subset T_{u(z)} E^h$. Since $J$ is horizontal and $u$ is $(j, J)$–holomorphic, projection of $Du$ must

give an injection from $T_zB$ onto the vertical tangent space $T_{u(z)}E^v$. Hence, by suitably choosing $Y$ at the single point $u(z)$, one can make $(Y \circ Du \circ j)_z$ any $(j, J)$–antilinear homomorphism $T_zB \longrightarrow T_{u(z)}E^v$. In particular, one can choose $Y$ at $u(z)$ such that $\langle \eta_z, (Y \circ Du \circ j)_z \rangle \neq 0$.

Reg$(u)$ is open, so making a smooth choice of $Y$ at nearby points of the section and applying a cut-off function contradicts the integral condition above. The condition that $z$ and nearby points are regular for $u$ means that there is no obstruction to extending such a choice of $Y$, defined at points of the section $u$, to a valid element of $\mathcal{T}_J^{h,G}$.

This is a contradiction unless $\eta = 0$ on $U \setminus \partial B$. However, by unique continuation $\eta = 0$ everywhere else. This achieves the desired contradiction, hence showing that $\widetilde{D}_{u,J}^{univ}$ surjects.

It now remains to prove the 2nd part of the Proposition. We shall prove that, under the given conditions, any horizontal $J$ is regular at all $u \in \mathcal{M}^h(Q)$.

The condition on symplectic curvature near the section $u$ causes the curvature of the symplectic connection to vanish, so gives a symplectic trivialisation of a neighbourhood containing $u$. Suppose $J$ also respected this trivialisation, then we could prove regularity by using the fact that, in the setting of Floer cohomology, $J$ is automatically regular at all constant holomorphic maps. Since this is not the case we use a generalisation of this idea by Seidel [2].

The symplectic triviality allows us to calculate the Maslov index $\mathrm{ind}(D_{u,J}) = 0$. Vanishing symplectic curvature also causes the action $A(u)$ of horizontal sections to vanish.

$$A(u) := \int_B u^*\Omega = 0$$

This gives us $\ker(D_{u,J}) = 0$ by [2, Lemma 2.27]. Hence $\mathrm{coker}\, D_{u,J} = 0$, and so $J$ is regular at $u$.

□

To apply the Proposition 2.45 to Floer cohomology one considers the case $G = \mathbb{R}$, acting by translations on the base $B = [0, 1] \times \mathbb{R}$. One could also consider a larger group with $\mathbb{R}$ as a quotient. When studying the relative invariant two applications are necessary. First the Proposition is applied to find regular equivariant (in particular translation invariant) almost complex structures over the ends. Then it is applied again, with $U \subset B$ disjoint from the ends, to find a regular almost complex structure on the fibration. This argument (applied with a modified equivariant regularity result) is used in Section 3.1.

# 3 A non-standard splitting of the relative invariant

In Lemma 2.40, I showed that the relative invariant of a fibre product splits as a product of the relative invariants of the factors. In this section I do the same for certain *twisted* products. These calculations are necessary in the setting of symplectic Khovanov homology to describe maps corresponding to particular presentations of trivial cobordisms (cf. stabilisation and destabilisation maps of Section 5.4).

The twisted products dealt with in this section are very simple cases of symplectic associated bundles. For similar, but more general details see the description of symplectic and holomorphic associated bundles at the end of Section 4.3 of [1].

Let $(X, \Omega_X = d\Theta_X, J_X)$ be an exact Kähler manifold. Suppose we have a holomorphic line bundle $\mathcal{F} \longrightarrow X$ which is a subbundle of a trivial bundle $\mathbb{C}^n \times X$. This gives us a straightforward choice of Hermitian metric on $\mathcal{F}$, defined by $||z||$ on $\mathbb{C}^n$. Define $\rho : \mathcal{F} \longrightarrow \mathbb{R}$ to be the restriction of $||z||^2$ to $\mathcal{F}$.

$\rho$ is plurisubharmonic on fibres and also $-\rho d(d\rho \circ i) = d\rho \wedge (d\rho \circ i)$. Consequently, the symplectic connection on $\mathcal{F}$, given by $-d(d\rho \circ i)$, has complex horizontal spaces $\ker(d\rho) \cap \ker(d\rho \circ i)$. The curvature of this connection is zero, so it induces local holomorphic trivialisations $\mathcal{F}|_U \longrightarrow \mathbb{C} \times U$, for simply connected $U \subset X$, which are respected by $\rho$. This means that the trivialisations restrict to trivialisations of the unit circle bundle $P := \rho^{-1}(1)$. Using these trivialisations all transition maps of $\mathcal{F}$, and hence also of $P$, must be of the form $(z, x) \longrightarrow (e^{i\theta} z, x)$ for constants $\theta$.

Now suppose we have another exact Kähler manifold $(E, \Omega_E = d\Theta_E, J_E)$, with a Hamiltonian $S^1$–action defined by the map $\mu : E \longrightarrow \mathbb{R}$. Assume also, that $J_E$ and $d\Theta_E$ are $S^1$–equivariant. Using the trivialisations of $P$ mentioned above one can form $P \times_{S^1} E$ and, by $S^1$–equivariance, assign it an exact Kähler structure $(\Omega = d\Theta, J)$ from those on $X$ and on $E$. It is important to note, in the case the $S^1$ action is part of a $J_E$–holomorphic $\mathbb{C}^*$–action, that $(P \times_{S^1} E, J)$ is canonically identified with the holomorphic associated bundle $(\mathcal{F} \setminus 0) \times_{\mathbb{C}^*} E$ (via the inclusion $P \times E \subset (\mathcal{F} \setminus 0) \times E$).

Other $S^1$–equivariant constructions can be lifted to $P \times_{S^1} E$. We shall be interested in the relative invariant in the case where $E$ is an MBL–fibration. As a warm-up I begin with the case of Floer cohomology. We shall implicitly always be requiring that Floer cohomology is well defined in each of the spaces for which we refer to it. It suffices for the applications later in this paper to make the assumptions of Remark 2.1 about the spaces involved.

For convenience, we shall make the assumption that $X, E$ are Stein, with exhausting plurisubharmonic functions $\rho_X, \rho_E$ respectively. All one really needs is some sort of holomorphic convexity.

**Lemma 3.1.** *Let $X$, $E$ be Stein manifolds, as above. Suppose we have compact exact Lagrangian submanifolds $L_A, L_B$, $K_A, K_B$ of $E$, $X$ respectively, and that $L_A, L_B$ are $S^1$–equivariant. Assume there exists a compatible $S^1$ equivariant almost complex structure $\tilde{J}_E$, which is regular at all holomorphic strips with boundary on $L_A, L_B$.*

*Then there is a canonical splitting of $HF^*(P|_{K_A} \times_{S^1} L_A, P|_{K_B} \times_{S^1} L_B)$ as the Künneth product of $HF^*(K_A, K_B)$ and $HF^*(L_A, L_B)$ (by this I mean tensor product at the level of chain complexes). This comes, under appropriate choice of almost complex structure from a canonical isomorphism of Floer cochain complexes*

$$CF^*(P|_{K_A} \times_{S^1} L_A, P|_{K_B} \times_{S^1} L_B) \cong CF^*(K_A, K_B) \otimes CF^*(L_A, L_B)$$

*Proof.* Let $\tilde{J}_X$, $\tilde{J}_E$ be regular, compatible almost complex structures on $X$, $E$ (with respect to the pairs of Lagrangians $L_A, L_B$, $K_A, K_B$) such that $\tilde{J}_E$ is also $S^1$-equivariant. Let $\tilde{J}$ be the induced almost complex structure on $P \times_{S^1} E$. Given any $(i, \tilde{J})$–holomorphic map $u : \mathbb{D} \longrightarrow P \times_{S^1} E$, projection to $X$ gives an $(i, \tilde{J}_X)$–holomorphic map $u_X : \mathbb{D} \longrightarrow X$. Projection to $E$ gives $(i, \tilde{J}_E)$–holomorphic maps $u_E : \mathbb{D} \longrightarrow E$, well-defined only up to the $S^1$–action on $E$.

Conversely, given holomorphic maps $u_X$, $u_E$ one defines a holomorphic map $u \longrightarrow P \times_{S^1} E$. Again this is well defined up to the $S^1$–action. Now we consider $u, u_X, u_E$ corresponding as described. By $S^1$ equivariance of the Lagrangians $L_A, L_B$ we have $u(z) \in P|_{K_A} \times_{S^1} L_A$ iff $(u_X(z), u_E(z)) \in K_A \times L_A$. The same holds for $K_B, L_B$, so we have a correspondence up to an $S^1$–ambiguity between holomorphic maps from $\mathbb{D}$ to $P \times_{S^1} E$ and to $X \times E$ with boundary on the appropriate Lagrangian submanifolds.

$\tilde{J}$ is regular at $u$ if and only if $\tilde{J}_X$ is regular at $u_X$ and $\tilde{J}_E$ is regular at $u_E$. This is because the $S^1$–equivariance of $\tilde{J}_E$ allows us to identify neighbourhoods of the holomorphic discs in $X \times E$ and $P \times_{S^1} E$. Since regularity at a particular section $u$ depends only on a

neighbourhood of the image of $u$, one can completely identify the deformation theories in the two cases. Furthermore, $\tilde{J}$ is compatible with $\Omega$ when $\tilde{J}_E$ and $\tilde{J}_X$ are compatible.

In conclusion, we have shown the following. There is an $S^1$–action (commuting with the usual $\mathbb{R}$–action) on the moduli spaces of holomorphic sections counted for the Floer differentials for $P|_{K_A} \times_{S^1} L_A, P|_{K_B} \times_{S^1} L_B$ inside $P \times_{S^1} E$ and for $K_A \times L_A, K_B \times L_B$ inside $X \times E$. The quotients by $S^1$ of these moduli spaces are identical, so in particular isolated ($\mathbb{R}$–orbits of) holomorphic strips must be fixed by the $S^1$–actions and so are in bijection. □

Now we generalise this to the case where $(E, \pi, \Omega_E, \Theta_E, J_0, \overline{\mathbb{D}}, j)$ is an exact MBL–fibration with the following properties:

- $E$ carries a fibrewise Hamiltonian $S^1$–action, given on any smooth fibre $\pi^{-1}(z)$ by a map $\mu|_{\pi^{-1}(z)}$ for some $\mu : E \longrightarrow \mathbb{R}$. In the Kähler region where $J_0$ is defined, the Hamiltonian flow of $\mu$ on the total space $E$ preserves fibres.
- $\Omega_E$ and $\Theta_E$ are $S^1$–equivariant, as is $J_0$ where it is defined.

In this case the same argument extends the splitting of Lemma 3.1 to split the relative invariant, given an $S^1$–equivariant regular compatible (relative to $j$) almost complex structure on $E$.

**Lemma 3.2.** *Let $X$ be a Stein manifold, as above. Let $E$ be an exact symplectic fibration together with an $S^1$–equivariant function $\rho_E : E \longrightarrow [0, \infty)$, which is exhausting and plurisubharmonic outside of some compact subset of $E$. Suppose we have exact Lagrangian submanifolds $K_A, K_B \subset X$ and an $S^1$–equivariant enclosed Lagrangian boundary condition $(Q_E, \rho_E, R_E)$ for $E$.*

*Assume also, that there is a compatible (relative to $j$) almost complex structure $\tilde{J}_E$ on $E$ which is $S^1$–equivariant and regular for $Q_E$ and which gives compatible, regular, time-dependent almost complex structures in the fibres over each end.*

*Then the relative invariant defined by $P \times_{S^1} E$ and $P|_{K_A} \times_{S^1} Q_E$, $P|_{K_B} \times_{S^1} Q_E$ over each half of $\partial\mathbb{D}$ respectively, splits as the identity map on $HF^*(K_A, K_B)$ and $\Phi_0^{rel}(E, Q)$ on the other factor. This comes, under appropriate choice of almost complex structure from a splitting on the cochain level. For the above, we use any large enough enclosure for the exact Lagrangian boundary condition on $P \times_{S^1} E$. Namely, $(\rho_E + \rho_X, R_E + R_X)$ suffices for large enough $R_X \in \mathbb{R}$.*

The proof is essentially the same as that for the previous Lemma.

**Remark 3.3.** One can relax the conditions that $X$ and the fibres of $E$ should be Stein. Then, given enclosures $(\rho_X, R_X), (\rho_E, R_E)$ on $X, E$ respectively, one would take an enclosure on $P \times_{S^1} E$ which is sufficiently $C^0$–close to $(\max\{\frac{\rho_E}{R_E}, \frac{\rho_X}{R_X}\}, 1)$.

The rest of this section is taken up with a particular calculation in a case where we are given $E$. The calculation is presented as an example application of the Lemmas above, for a specific case required relevant to Section 5.

## 3.1 Finding a regular, equivariant almost complex structure

In sections 3.1 to 3.3 we show the existence of an $S^1$–equivariant almost complex structure for a specific exact MBL–fibration (actually a Lefschetz fibration) needed for the study of stabilisation and destabilisation maps and use it to calculate the relative invariant. No attempt is made here to achieve the most general results possible.

Consider the following singular symplectic fibration for fixed $d \in \mathbb{C}^*$ (see Lemma 4.6 for motivation):

$$\begin{array}{ccc} \mathbb{C}^3 & & (a,b,c) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathbb{C} & & a^3 - ad + bc \end{array}$$

Define $\rho_{std} := |a|^2 + |b|^2 + |c|^2$. We take the exact symplectic form $\omega_{std} = d\theta_{std} = -d(d\rho_{std} \circ i)$ on $\mathbb{C}^3$ together with the standard complex structure $i$.

Choose $x \in \mathbb{C}$, a square root of $\frac{d}{3}$. There is a single critical point in the fibres over each of $\pm 2x^3$. There is also an important Hamiltonian $S^1$–action $(a,b,c) \mapsto (a, \zeta b, \zeta^{-1} c)$ generated by the Hamiltonian $\mu(a,b,c) = |b|^2 - |c|^2$. Out of this fibration we will construct an exact MBL–fibration whilst maintaining the $S^1$–equivariance of $\theta$.

We now restrict the fibration to a small holomorphic disc $B \subset \mathbb{C}$, of radius $\epsilon$, containing only the singular value $-2x^3$. Call this new fibration $E$. This disc may have any number of boundary marked points, though we are interested mostly in the case where it has 2. We also 'flatten' the fibration using Lemma A.1 and deform it near the boundary marked points to give $B$ striplike ends (see Proposition 2.22). Both deformations can straightforwardly be chosen to preserve $S^1$-equivariance and both also preserve non-negativity of the symplectic curvature. Denote the resulting exact MBL–fibration by $(E, \pi, \Omega, \Theta, J_0, B, j)$.

$E$ carries some further structure, namely the (pullback of the) projection $p$ to the $a$–coordinate, which restricts to a Lefschetz fibration on each regular fibre of $\pi$. Each of these has three critical points over distinct values of $a$. As one approaches a critical fibre of $\pi$ two of these collide. By inspection, the $S^1$–action is also fibrewise w.r.t. $p$. Hence, it preserves symplectic parallel transport vector fields and the symplectic parallel transport over $\pi$ and $p$ preserves $\mu$. This gives us control of vanishing and matching cycles in the following manner.

**Lemma 3.4.** *Any $S^1$–equivariant Lagrangian sphere in a regular fibre of $E$ (as above) can be expressed as $\mu|_{E_z}^{-1}(0) \cap p|_{E_z}^{-1} \operatorname{im} \gamma$ for some smooth embedded path $\gamma$ in $\mathbb{C}$ between critical values of $p|_{E_z}$. Furthermore, the vanishing cycle in for any path in $B$ into the critical value of $\pi$ is of this form. In this case $\gamma$ is a smooth path between the two critical values of $p|_{E_z}$ which collide in the critical fibre of $\pi$.*

*Proof.* Denote the fibre $p|_{E_z}^{-1}(a)$ over $a \in \mathbb{C}$ by $E_{z,a}$. Consider a Lagrangian $L \subset E_z$. By $S^1$–equivariance, $L$ intersects fibres $E_{z,a}$ in possibly empty unions of circles, except at critical points of $\mu$ (which are precisely the critical points of $p|_{E_z}$). At any regular $x \in L$ the intersection $T_xL \cap (T_xE_{z,a})^{\perp\Omega}$ is one-dimensional so we can define a non-vanishing vector field locally on $TL$ of these vectors. Projecting these to $\mathbb{C}$ by $Dp$ makes them all tangent, since otherwise we reach a contradiction with $S^1$–equivariance and the dimension of $L$.

Hence, locally $L$ is preserved by symplectic parallel transport over certain paths in $\mathbb{C}$. The only way for components of $L$ to be closed spheres is for these paths to end at distinct critical points.

All the critical points satisfy $\mu = 0$, and $\mu$ is preserved by symplectic parallel transport, so it suffices to observe that the $S^1$–action is transitive on any set of the form $\mu|_{E_{z,a}}^{-1}(0)$ (these are circles or points). $\square$

Under certain conditions on $(E, \pi)$ and $Q$ we can use Proposition 2.45 to choose an $S^1$–equivariant regular almost complex structure given any choice of compatible $S^1$–equivariant

almost complex structure in the fibres over the ends. It is in fact more difficult to choose the almost complex structures in the fibres over the ends to be regular (which is necessary for the relative invariant to make sense as a map of Floer cochain complexes).

**Lemma 3.5.** *Let $(E, \pi)$ be as above with any $S^1$–equivariant, transverse, enclosed exact Lagrangian boundary condition $Q$. Assuming:*

- *All $S^1$-equivariant sections with boundary on $Q$ are horizontal and contained in a region of $E$ with symplectic curvature identically zero*
- *We have chosen $S^1$–equivariant, time-dependent almost complex structures $J_e$ over each end*

*Then there exists a regular, $S^1$–equivariant (horizontal) almost complex structure (in the set $\mathcal{J}^{S^1}(E, \pi, \{J_e\}, Q)$).*

**Remark 3.6.** To apply this to our particular situation we will need to show that the boundary condition $Q$ can be chosen to be transverse whilst maintaining $S^1$–equivariance. In particular, this will mean that the Lagrangian intersections given by $Q$ in fibres over ends must be fixed points of the $S^1$–action.

It is also necessary for the radial vanishing locus to the singular point of $E$ not to hit the intersection of the exact Lagrangians over at least one of the ends. For further discussion of this, see section 3.2.

*Proof.* We apply Proposition 2.45. This requires that we check various conditions.

It suffices to take $U$ to be the entire base minus the closure of a sufficiently small open neighbourhood of each end. In particular, the symplectic curvature should be zero away from $\pi^{-1}(U)$. Take any partial horizontal section $w$ over $U$ with boundary in $Q|_U$. Restricting to a striplike end and projecting to the fibre at infinity we must get a constant map to an intersection point of our Lagrangians there. In particular $w$ is $S^1$–equivariant there. $\Omega$ is also $S^1$–equivariant and so symplectic parallel transport preserves the locus of $S^1$–fixed points (which is isolated in each fibre). This means that either:

- $w$ extends to a horizontal section through the symplectically flat trivialised region of $E$ (away from the radial vanishing locus)
- or $w$ contains one of the $S^1$–fixed points in a fibre which is carried to the singular point by radial symplectic parallel transport. However, this cannot happen since then $w$ would contain the singular point of $E$ at which no section can be smooth.

Given any non-horizontal, holomorphic section for any $J \in \mathcal{J}^{h,S^1}(E, \pi)$ unique continuation means that the set of points in the base which don't map to $S^1$–fixed points of $E$ is open and dense. This is by the same argument as above.

The conditions for regularity of $J$ at horizontal sections are also clearly satisfied. ☐

We now prove the more difficult result, that the almost complex structures over the ends can be chosen to be regular for the calculation of Floer cohomology, namely:

**Proposition 3.7.** *Let $(M, \Omega, \Theta, \rho, J)$ be a Stein 4–manifold carrying a Hamiltonian $S^1$–action which is free on some orbit and is given by the Hamiltonian $\mu$. Let $p \colon M \longrightarrow \mathbb{C}$ be an $S^1$–equivariant Lefschetz fibration and $L_0$, $L_1$ be a pair of transverse $S^1$–equivariant exact Lagrangian spheres contained in the set $\mu^{-1}(0)$.*

*Then there is a regular $S^1$-equivariant almost complex structure $J' \in \mathcal{J}^{S^1,reg}(M, L_0, L_1)$ arbitrarily $C^\infty$–close to $J$. In particular, $J' \in \mathcal{J}^{S^1,reg}(M, L_0, L_1)$ is non-empty.*

*Proof.* We can view this problem equivalently in terms of horizontal almost complex structures on the trivial exact MBL–fibration $M \times [0,1] \times \mathbb{R}$ over the infinite strip, which are also equivariant under the $\mathbb{R}$ translation action in the base. It is, however, necessary to refine the argument of Proposition 2.45. We restrict attention to the set of $S^1 \times \mathbb{R}$–equivariant horizontal almost complex structures which also make the projection $p$ to $\mathbb{C}$ holomorphic for some complex structure $i_t$ on $\mathbb{C}$ depending on the 'time coordinate' $t \in [0,1]$ of the base. In the following argument we show that a generic perturbation of $J$ within this set of almost complex structures is regular.

We take the set $U \subset B$ (as in Proposition 2.45), over which the almost complex structure is to be perturbed, to be the entire base. Let $(u, \tilde{J})$ be a non-horizontal smooth section together with $S^1 \times \mathbb{R}$–equivariant horizontal almost complex structure making $u$ $(j, \tilde{J})$–holomorphic and making $p$ $(\tilde{J}, i_t)$–holomorphic. There are two necessary steps to adjust the proof of Proposition 2.45 to this case. Namely, there should exist a non-empty open subset $\tilde{U}$ of the base $[0,1] \times \mathbb{R}$ such that:

(1) The set of regular points $\mathrm{Reg}(u)$ of $u$ contains $\tilde{U}$.

(2) One can choose tangents $Y$ to the space of such $\tilde{J}$ at individual points $u(z)$ over $\tilde{U}$ such that $(Y \circ Du \circ j)_z$ gives us any $(j, \tilde{J})$–antilinear homomorphism $T_zB \longrightarrow T_{u(z)}M$ (after projection to $TM$).

The same proof by contradiction argument as for Proposition 2.45 then works, choosing a first order deformation $Y$ supported near $u(\tilde{U})$.

We start by proving (1). Consider the projection $p \circ u$ and the curves $\gamma_0, \gamma_1$ to which $p$ maps $L_0, L_1$ respectively. By adding 2 points to $[0,1] \times \mathbb{R}$ this becomes a map of $\overline{\mathbb{D}}$ to $\mathbb{C}$ with one half of the boundary mapping to each of $\gamma_0, \gamma_1$. Since the Lagrangians are transverse, they intersect in isolated points, so $\gamma_0, \gamma_1$ intersect only at their endpoints. Hence, either $p \circ u$ is homotopic to the constant map to an intersection of $\gamma_0$ with $\gamma_1$, or $\gamma_0, \gamma_1$ share both endpoints (and form the boundary of a simply connected region of $\mathbb{C}$) in which case $p \circ u$ is homotopic to a bijection onto this region.

Two properties derived from the $(j, i_t)$ holomorphicity of $p \circ u$ are necessary in the following. Firstly, $p \circ u$ must be nowhere orientation reversing. Secondly, direct consideration of the Cauchy Riemann equations implies that $p \circ u$ is either constant or positively covers some area in $\mathbb{C}$.

In the case that $p \circ u$ is homotopic to the constant map, the integral of the pullback of the standard area form from $\mathbb{C}$ must vanish, so we conclude $p \circ u$ is a constant map. This contradicts the assumption that $u$ is not horizontal.

Hence the integral of the pullback of the standard area form from $\mathbb{C}$ must be equal to the area surrounded by $\gamma_0, \gamma_1$. By Sard's theorem there exist regular values of $p \circ u$ in this region. By the Brouwer Fixed point theorem these are all in the image. Hence there is some non-empty open set $\tilde{U} \subset [0,1] \times \mathbb{R}$ which embeds onto its image under $p \circ u$. Furthermore, $p \circ u(z) \notin p \circ u(\tilde{U})$ for $z \notin \tilde{U}$. Otherwise, the nowhere orientation reversing property would contradict the fact that $p \circ u$ is homotopic to a bijection.

The actions of $S^1$ and $\mathbb{R}$ both fix $p$ and the $S^1$–action is free at all points of $u(\tilde{U})$, so we have now shown that $\tilde{U}$ is contained in the set of regular points of $u$.

In order to prove (2) we first require w.l.o.g. that $p \circ u(\tilde{U})$ is disjoint from an open neighbourhood of the singular values of $p$. This means that we can split $TM \oplus TB$ at every point of $u(\tilde{U})$ as

$$\ker\,(Dp) \oplus (\ker Dp)^{\perp_\Omega} \oplus TB$$

In this splitting the complex structures we consider can be written at $u(z)$ as:

$$\tilde{J}_z = \begin{pmatrix} R & S & \\ & i_t & \\ & & j \end{pmatrix}$$

Here $R$ is a linear complex structure on $\ker Dp$ at any point and $RS + Si_t = 0$.

A generic tangent $Y$ to this space of complex structures, at points of $u(\tilde{U})$, has the form:

$$Y_z = \begin{pmatrix} A & B & \\ & C & \\ & & 0 \end{pmatrix}$$

but it will suffice to restrict our attention to the case where $A = 0$. The only other restrictions on $Y$ are $Ci_t + i_t C = 0$ and $Bi_t + RB + SC = 0$.

Similarly w.r.t. the same splitting $Du$ is written:

$$Du = \begin{pmatrix} F \\ G \\ I \end{pmatrix}$$

Here $G$ is complex $(j, i_t)$-linear and $RF + SG = Fj$. Since $u$ is non-horizontal, unique extension implies that $G$ is an isomorphism for an open dense set of choices of $z$ in the base. Hence, we may assume w.l.o.g. that this is the case for all $z \in \tilde{U}$. This means, by choosing $i_t$–antilinear $C$, we can make $CG$ any $i_t$–antilinear map we like. Furthermore, the condition on $B$ is invariant under right multiplication of $B$ and $C$ with $G$ and with $G^{-1}$, so we may choose the $Y \circ Du$ at $u(z)$ and hence also $Y \circ Du \circ j$ to give any complex antilinear homomorphism $T_z B \longrightarrow T_{u(z)} M$. □

## 3.2 Transversality of equivariant Lagrangian boundary conditions

Let $(E, \pi)$ be the fibration defined at the beginning of Section 3.1. We show in this section, that two particular Lagrangian boundary conditions can be chosen to be transverse without losing $S^1$–equivariance.

These Lagrangian boundary conditions are related to stabilisation and destabilisation of bridge diagrams (see Section 5.4). Given the description of such Lagrangians in any fibre by lines in the plane (Lemma 3.4), this means choosing a Lagrangian boundary condition $Q$ such that these lines (defined by Lagrangians in the fibre over any end) intersect transversely and only at their endpoints. For purely topological reasons relating to the isotopy classes of these lines, one can in general not rule out intersections along the interior of the lines (corresponding to clean intersections along copies of $S^1$). However, we study two cases where the topological obstructions vanish.

By Lemma 3.4 any $S^1$–equivariant $Q$ lies entirely in $\mu^{-1}(0)$. Furthermore the $S^1$–action is transitive on fibres of $p|_{E_z \cap \mu^{-1}(0)}$ for any $z$. This means

(a) Symplectic parallel transport restricted to $\mu^{-1}(0)$ gives a well-defined flow in the plane by composing the time–$t$ maps with $p$.

(b) Hamiltonian flows on any $E_z$ with Hamiltonian factoring through $p$ similarly project under $p$ to well-defined flows in the plane.

(c) On the $\mu = 0$ locus, restricting $\Omega$ to $\ker Dp^{\perp_\Omega}$ and pulling back by $Dp^{-1}$ gives a well defined symplectic form on $\mathbb{C}$ away from the critical values of $p|_{E_z}$ such that the Hamiltonian flow of any $h \circ p$ on $E_z$ projects to the Hamiltonian flow of $h$ on $\mathbb{C}$.

(b) and (c) above allow us to choose the Lagrangian boundary conditions $Q^1_+$, $Q^0_+$ over one marked point by isotopy of their projections in the plane (fixing endpoints). Then, using (a), the projections of $Q^1_-$, $Q^0_-$ are given by applying a flow in the plane coming from the symplectic parallel transport (around $E$ on the $\mu = 0$ locus), projected by $p$.

For $x$ small and the radius $\epsilon$ of the base $B \subset \mathbb{C}$ much smaller (see definition of $(E, \pi)$ in Section 3.1), the projection $p$ on any fibre has two critical values $q, q'$ near $x$ and one near $-2x$. The vanishing cycle for the critical point of $\pi$, taken in any non-singular fibre, projects to a curve $\alpha$ between $q$ and $q'$. For fibres near enough to the singular fibre (and vanishing cycle construction performed over straight line segment to the singular value of $\pi$), this is approximately a line segment.

We shall consider $S^1$–equivariant $Q$ such that the projections of $Q^1_+$, $Q^0_+$ are as given in Figure 7. This particular setup will later be of relevance to the stabilisation map. The corresponding calculations for other settings are essentially the same, so we shall only go into detail with this calculation.

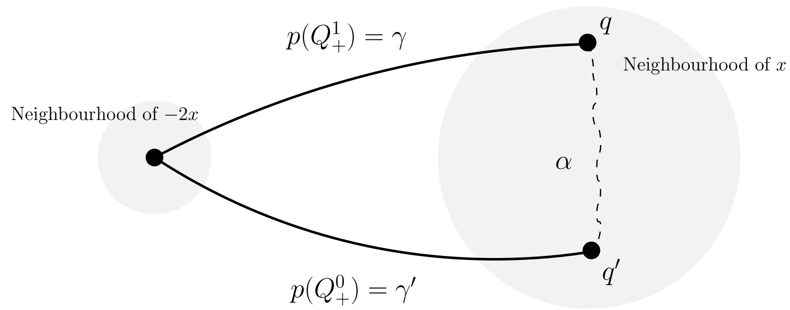


Figure 7: Projections of Lagrangians to curves $\gamma$, $\gamma'$ with a single transverse intersection at an endpoint. This describes the Lagrangians at the output end of the fibration.

The monodromy (anticlockwise) around the boundary of $(E, \pi)$ is a Dehn twist in the equivariant Lagrangian sphere projecting to the curve $\alpha$. This is achieved on the $\mu = 0$ locus by an anticlockwise half twist in $\gamma$. Hence the projections of $Q^0_-$ and $Q^1_-$ **up to isotopy** are as illustrated in Figure 8.

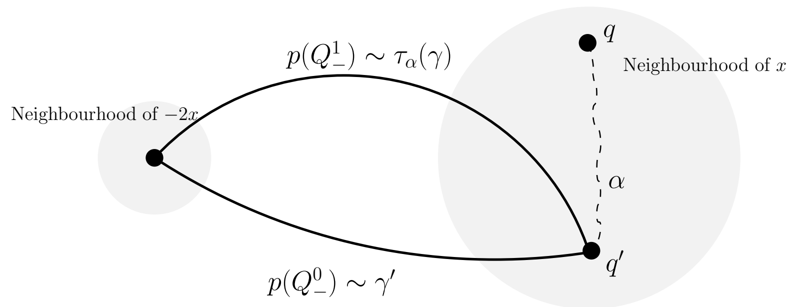


Figure 8: The curves of Figure 7 after a half twist $\tau_\alpha$ in the curve $\alpha$ has been applied to $\gamma$. This describes the Lagrangians at the input end of the fibration.

However, we require that the projections actually intersect no more than in the diagram.

**Lemma 3.8.** *With appropriate choices in the definition of $(E,\pi)$ and choices of $Q^1_+$, $Q^0_+$ we may ensure that $Q^0_-$, $Q^1_-$ project to curves intersecting transversely and only at their end points.*

We first reduce this Lemma to one about curves in the plane under a specific flow.

Let us consider in detail the construction of $E$. One first pulls back the fibration $\pi : \mathbb{C}^3 \longrightarrow \mathbb{C}$ by a map from $B$ onto a small enough disc in $\mathbb{C}$, then deforms the one form $\Theta$ to $\tilde{\Theta} = \Theta - \psi\beta r^2 d\alpha$ (see Appendix A). The deformation is supported outside of a neighbourhood $U$ of the radial vanishing locus of $E$ and removes all symplectic curvature outside a larger neighbourhood $U'$.

Consider the definition of $\psi$. It is first defined in the singular fibre, then extended by radial symplectic parallel transport. In this singular fibre we may choose it in a neighbourhood of $\mu^{-1}(0)$ to depend only on the $a$–coordinate. This has the effect that in a neighbourhood of $\mu^{-1}(0)$ in any fibre $\psi$ also depends only on the $a$–coordinate.

We may choose $\psi = p^* k_0$ in the singular fibre for some function $k_0 \colon \mathbb{C} \longrightarrow [0,1]$ with the following properties (in terms of a constant $\delta \ll |x|$ to be defined later):

- $k_0 = 0$ within radius $2\delta$ of $x$
- $k_0 = 1$ outside radius $3\delta$ of $x$

$\psi$ is then extended to other fibres by radial symplectic parallel transport restricted to $\mu^{-1}(0)$. This gives in particular a new function $k_z$ on each fibre $\pi^{-1}(z)$, such that $\psi|_{\pi^{-1}(z)} = p^* k_z$.

The radial symplectic parallel transport, the symplectic parallel transport before the flattening deformation and the symplectic parallel transport where $\psi$ is identically zero are still that of the original fibration. A short calculation reveals that the flow given by symplectic parallel transport on the $\mu = 0$ locus of the fibration $\mathbb{C}^3 \longrightarrow \mathbb{C}$ (with standard symplectic form on $\mathbb{C}^3$) over a path $z(t)$ in the base is given, independently of $b, c$, by the vector field

$$\frac{\partial}{\partial t} a = \frac{3\overline{(a^2 - x^2)}}{9\left|a^2 - x^2\right|^2 + 2\left|z + 3ax^2 - a^3\right|} \dot{z}(t)$$

For convenience we choose $x$ to be positive, real and small. Fixing any time $t$, and consequently fixing $z(t)$, the flow always has a non-degenerate critical point at $a = x$. Also the flow lines depend only on $a$ and the argument of $\dot{z}$. From the formula above one sees that, by choosing $\delta$ small enough, the directed flow lines can be made

- $C^1$–close to being linear on a $2\delta$–neighbourhood of $-2x$
- $C^1$–close to flow lines for the flow $\frac{\partial}{\partial t} a = \frac{2}{3}\overline{(a - x)}\dot{z}$ on a $2\delta$–neighbourhood of $x$

At $x$ the unstable manifold has argument $\frac{1}{2}\arg(\dot{z})$ and the stable manifold argument $\frac{\pi}{2} + \frac{1}{2}\arg(\dot{z})$.

We consider restricting the base of the fibration to a very small $\epsilon$–neighbourhood of the singular value $-2x^3$ (compare this to a similar restriction in Appendix A). In fact we will restrict the base of the fibration to a small isosceles triangle within the $\epsilon$–neighbourhood, symmetric over the real axis with a vertex $v$ of angle $4\phi$ on the real line to the right of $-2x^3$ (see Figure 9). For small $\phi$ this can be done such that the triangle contains the disc of radius $\epsilon \sin(2\phi)$ around $-2x^3$. We will shortly choose a sufficiently small value for $\epsilon$.

The speed of the flow in the $a$–coordinate outside of $\delta$–neighbourhoods of $\pm x$ is now bounded above:

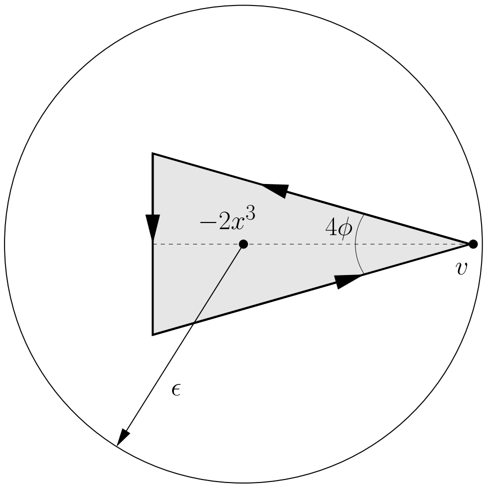


Figure 9: Image of the base $B$ of the fibration $E$ in $\mathbb{C}$

$$\left|\frac{\partial}{\partial t}a\right| \leq \frac{3\left|a^2-x^2\right|\epsilon}{9\left|a^2-x^2\right|^2+2\left|z+3ax^2-a^3\right|} < \frac{|\dot{z}|}{3\left|a^2-x^2\right|} < \frac{|\dot{z}|}{3\delta^2}$$

This gives us the first condition on a choice of $\epsilon$. Radial symplectic parallel transport between the singular value and any point on the boundary of the restricted base should move points with $|a-x| \in [2\delta, 3\delta]$ a distance of at most $\frac{\delta}{2}$. For example it suffices to take $\epsilon \leq \frac{3}{4}\delta^3$.

From this we see that in any boundary fibre the function $k$ remains identically 0 inside a $\delta$–neighbourhood of $x$ and identically 1 between a $4\delta$–neighbourhood of $x$ and a large ball around 0. Considering the diffeomorphisms which deform $k_0$ to $k_z$ away from the singular fibre, a compactness argument gives us some constant $C$ such that on $\mathbb{C}$ we have for all $z$ that $|dk_z| \leq C$. Hence on any vector tangent $V \in \ker D\pi$ to a fibre of $\pi$:

$$|d\psi(V)| = |d(p^*k_z)V| \leq |dk|\,|Dp(V)| \leq |dk|\,|V|$$

We now consider the symplectic parallel transport around the boundary of the restricted base. Let $\dot{z}$ be a constant speed, unit time (piecewise smooth) parametrisation of this boundary. We require $\epsilon$ be small enough that, before the deformation of the symplectic structure, points outside of $\delta$–neighbourhoods of $\pm x$ move with bounded speed

$$\left|\frac{\partial}{\partial t}a\right| \leq \frac{\delta}{2}$$

Here it suffices to take $\epsilon \leq \frac{3}{4\pi}\delta$.

The deformed symplectic form (cf. Appendix A) is $\tilde{\Omega} = d\Theta - \psi\beta r^2 d\alpha$ where $(r,\alpha) = \left(\left|z+2x^3\right|, \arg\left(z+2x^3\right)\right)$. With respect to the metric arising from $\Omega$ and the standard complex structure, we find by compactness that there is a bound (w.l.o.g. $C$) on $\beta$ and $|d\beta|$ on the compact region of $\mu^{-1}(0)$ with $|a-x| \in [\delta, 4\delta]$. This bound is taken before restriction of the base to an $\epsilon$–neighbourhood of $-2x^3$.

Over the boundary path $\dot{z}$ of the restricted base (with speed at most $2\pi\epsilon$) let $H, V+H$ be the symplectic parallel transport vectors of $\Omega$, $\tilde{\Omega}$ respectively over some $\dot{z}$. Then $V \in \ker D\pi$ and for all unit norm (w.r.t. $\Omega$ and the standard complex structure) vectors $W \in \ker D\pi$

$$
\begin{aligned}
0 &= \tilde{\Omega}(V+H,W) \\
&= \Omega(V+H,W) - d(\psi\beta r^2)\wedge d\alpha(V+H,W) \\
&= \Omega(V,W) + r^2(\psi d\beta + \beta d\psi)(W)d\alpha(\dot{z})
\end{aligned}
$$

We have w.l.o.g. $|d\alpha(\dot{z})| \leq \frac{2\pi}{\sin 2\phi} \leq C$, so:

$$||V+H|| \leq ||Dp(V)|| + 2\pi\epsilon \leq 2\epsilon^2 C^3$$

and for small enough $\epsilon$ this is bounded above by $\frac{\delta}{2}$.

Given this control of the symplectic parallel transport away from $a = x$ it remains to establish some control in the $\frac{3}{2}\delta$ neighbourhood of $x$. In this region $\tilde{\Omega} = \Omega$, so we have an explicit formula for $\frac{\partial}{\partial t}a$.

The transversality result locally in the $\frac{3}{2}\delta$–neighbourhood of $x$ is as follows. One should note, the previous argument should actually be run at the same time as this one to define $\delta, \epsilon$ appropriately. However, it can be checked that neither obstructs the other.

**Lemma 3.9.** *For $\epsilon$ small enough, the two critical values $q$ and $q'$ of the map $p$ are contained in the $\delta$–neighbourhood of $x$. It is possible to choose the curves $\gamma = p(Q^1_+)$, $\gamma' = p(Q^0_+)$ in the isotopy classes indicated in Figure 7 such that:*

- *$\gamma$, $\gamma'$ are disjoint except at one end-point.*
- *within the $\frac{3}{2}\delta$–neighbourhood $\tau_\alpha(\gamma)$ and $\gamma'$ intersect transversely and only at their end-points.*

*Proof.* For $z = v$ the critical values $q$, $q'$ lie on the real line on either side of $x$. We divide the monodromy into three stages, corresponding to the sides of the triangle (Figure 9). In the first stage the flow lines are ($C^1$–close to) those in Figure 10 (left). We will denote the curve $\gamma$ after each stage of the monodromy by $\gamma_1, \gamma_2, \gamma_3 = \gamma'$.

We choose $\gamma$ to be one of the flow lines ending at a $q$ (up to a sufficient distance outside the $\frac{3}{2}\delta$–neighbourhood of $x$, outside of which it joins to the third root). This means that $\gamma$ remains confined to the flow line for this entire stage of the monodromy. In particular $\gamma_1$ is confined to a single quadrant (with boundaries on the stable and unstable flow lines).

The total monodromy, up to isotopy, is a positive half twist swapping $q$ and $q'$. Combine this with the fact, that during each stage points cannot cross flow lines and one has a lower bound on the increase in argument of $q - x$ after each stage. After the first stage $q$ must move to a point in $q_1$ the shaded region.

Let $l$ be the flow line (in the second stage) through the endpoint of $\gamma_1$. Flow lines from the first two stages intersect each other in at most one point, so $\gamma_2$ lies entirely to one side of $l$. Furthermore, $\gamma_1$ cannot intersect the unstable manifold (from the second stage) so $\gamma_2$ cannot either. Hence $\gamma_2$ lies in the shaded region of Figure 10 (right).

Since $\gamma_2$ is confined to this region with its end on the boundary we can now choose another curve $\tilde{\gamma}$ from the third root to the endpoint of $\gamma_2$ which is otherwise disjoint from $\gamma_2$ in the $\frac{3}{2}\delta$–neighbourhood of $x$. Applying the monodromy of the third stage to $\tilde{\gamma}$ gives us $\gamma'$ as required. ☐

One now combines this Lemma with the restriction on $\left|\frac{\partial}{\partial t}a\right|$ to prove Lemma 3.8. The following observations suffice to make the required choice of $\gamma, \gamma'$ possible.

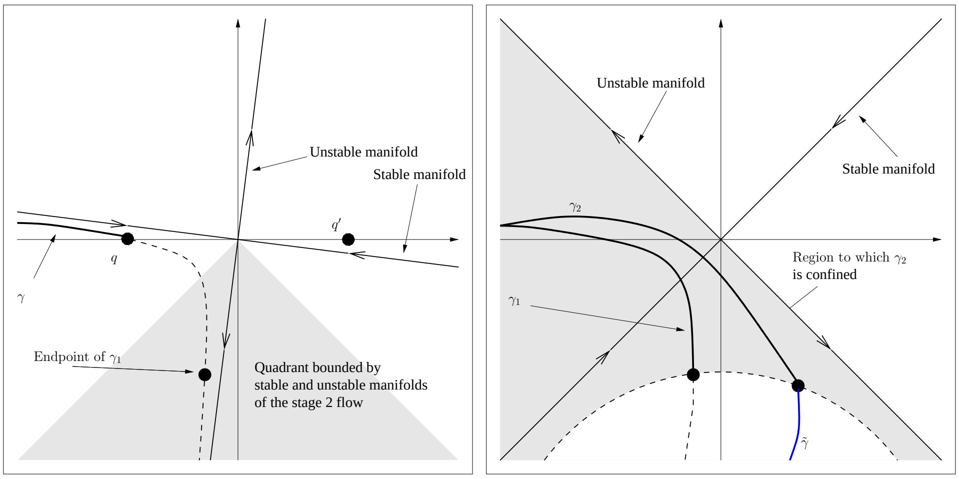


Figure 10: The flow as $z$ traverses the first (left hand diagram) and second (right hand diagram) side of the triangle anticlockwise from $v$

- In a $\frac{3}{2}\delta$–neighbourhood of $x$ we have the local result above.
- Using the trivialisation of the symplectic fibration with 2–form $\tilde{\Omega}$. we see that the monodromy around the entire boundary of $B$ is zero for points in $\mu^{-1}(0)$ with $a$–coordinate outside of a $4\delta$–neighbourhood of $x$
- The monodromy around the entire boundary of $B$ moves points outside of $\frac{3}{2}\delta$–neighbourhood of $x$ a distance of at most $\frac{\delta}{2}$.

### 3.3 Two calculations of the relative invariant

In this Section we calculate up to sign the relative invariant in two cases which fit the framework of Sections 3.1 and 2.4. These are the two cases needed in Section 5.4 to understand the stabilisation and destabilisation maps.

We use the fibration $(E, \pi)$ defined in the beginning of Section 3.1. By Lemma 3.8 we may choose the Lagrangian boundary condition $Q$ to be both transverse and $S^1$–equivariant. This means $Q$ is contained in the $\mu = 0$ locus and can be represented in any fibre by a line in $\mathbb{C}$ between critical values of $p$.

The existence of a regular, $S^1$–equivariant almost complex structure (by Lemma 3.5 and Proposition 3.7) reduces the calculation of the map induced on Floer cohomology by $E$ to simple combinatorics related to curves in the plane. This is because the $S^1$–action on $E$ induces a corresponding action on the moduli spaces of holomorphic sections. Hence, isolated holomorphic sections are precisely those fixed by the $S^1$–action.

There are in any regular fibre of $E$ precisely three fixed-points of the $S^1$–action, namely the critical points of $p$. Two of these cannot be used in any such section since they would force the section to pass through the critical point of the MBL–fibration $E$. This leaves the section formed by the third point. In the setting of Figures 7 and 8, this point corresponds to the critical point of $p$ with $a$–coordinate near to $-2x$ and is always in both Lagrangians.

The map of chain complexes has now been reduced to counting a single isolated holomorphic section.

The chain complexes used will be:

- $\mathbb{Z}$ (with the only possible relative grading) for the case of two Lagrangians meeting transversely at a single point as in Figure 7. The differential can only be trivial.
- $\mathbb{Z}^2$ when the Lagrangians meet at two points and are Hamiltonian isotopic as in Figure 7. In this case the cohomology is $H^*(S^2)$, so the differential is also trivial. We will write this as $\mathbb{Z}[X]/(X^2)$ where $X$ has degree 2 greater than 1.

In the latter case, one can see which intersection represents $X$ by relating the relative Maslov and relative Morse indices. Namely, the singular point of $p$ near $-2x$ represents 1 and $q'$ represents $X$.

The calculations of the two maps illustrated in Figure 11 are now immediate.

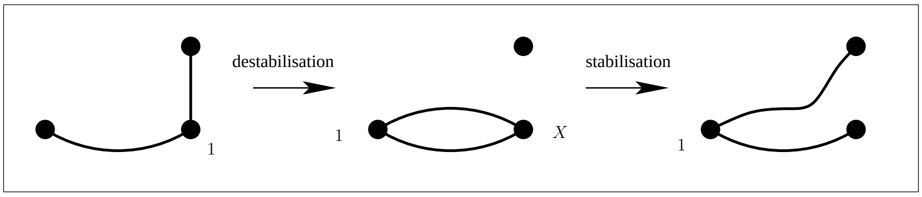


Figure 11: An illustration of the projections under the map $p$ of the $S^1$–equivariant Lagrangians for the two particular relative invariants. These correspond, as labelled, to parts of the stabilisation and destabilisation maps for symplectic Khovanov homology (see Section 5.4). Generators of the chain complexes for Floer cohomology are labelled.

**Lemma 3.10.** *The relative invariants illustrated in Figure 11 are, up to an overall sign ambiguity, given by:*

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}[X]/(X^2) \\ 1 & \mapsto & X \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{Z}[X]/(X^2) & \longrightarrow & \mathbb{Z} \\ X & \mapsto & 0 \\ 1 & \mapsto & 1 \end{array}$$

# 4 Symplectic Khovanov homology

## 4.1 A short summary of symplectic Khovanov homology

This section summarises the construction of $\mathrm{KH}_{\mathrm{symp}}$ with emphasis on those parts of most relevance later. The construction is based on the observation that braids on $n$ strands can be viewed as loops in the configuration space $\mathrm{Conf}_n(\mathbb{C})$ of $n$ points in the plane.

Suppose we have a symplectic fibration $\chi : S \longrightarrow \mathrm{Conf}_n(\mathbb{C})$ on which symplectic parallel transport maps are well-defined. Then symplectic parallel transport defines a map

$$\pi_1(\mathrm{Conf}^0_{\mathrm{n}}(\mathbb{C}), P) \longrightarrow \pi_0(\mathrm{Symp}(\chi^{-1}(P)))$$

for any base point $P \in \mathrm{Conf}_n(\mathbb{C})$. This is a representation of the braid group. Ideally one would also require the symplectic parallel transport maps to be compactly supported.

This sort of symplectic geometry is not new. Khovanov and Seidel [16] show that similar symplectic geometry can be used to construct Khovanov's categorification of the Burau representation of the braid group.

We use instead a fibration with subtly different properties. Let $\mathrm{Conf}^0_{\mathrm{n}}(\mathbb{C})$ be the space of configurations with coordinates summing to 0 and let $\overline{\mathrm{Conf}}^0_n(\mathbb{C}) \cong \mathbb{C}^{n-1}$ be its closure in

$\mathrm{Sym}_n(\mathbb{C})$. Seidel and Smith [1] define a singular holomorphic fibration of a Stein manifold $S_n$ over $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$. $S_n$ has the property that it pulls back to an exact MBL–fibration over any holomorphic discs in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ passing transversely through the locus of $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})\backslash\mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$ where precisely two coordinates meet (cf. Lemma 27 of [1]).

Let $\overline{\mathbb{D}}$ be such a disc with a single singular fibre and let $C$ be a bounded subset of the singular locus of the pullback fibration over $\overline{\mathbb{D}}$. Then, for simple enough paths $\gamma$ (short and linear in $\overline{\mathbb{D}}$ suffices), the vanishing locus $V_C$ to $C$, in nearby regular fibres over $\gamma$, is well defined (see [1, Section 4.2]). We refer to $V_C$ as the relative vanishing cycle to $C$. If $C$ is open, then $V_C$ is a bounded coisotropic, fibring over $C$ with fibres all isotropic spheres.

The monodromy of the elementary braid performing a single positive twist in the two chosen coordinates is realised by a loop around the singular value of $\overline{\mathbb{D}}$. If one is careful to make the loop small, then the monodromy map is well defined over an open neighbourhood of $V_C$. By a deformation of the exact symplectic structure, the monodromy can be taken to restrict to a neighbourhood of $V_C$ as a fibred Dehn twist in the case that $C$ is open. This is a consequence of work of Perutz [18]. Non-compactness of the singular locus and of $V_C$ are a problem here, so we shall not make direct use of this description, however, it is useful heuristically.

The path into the singular value of $\overline{\mathbb{D}}$ corresponds naturally to a 'singular braid' of the form of the $(n-2,n)$–tangle in Figure 12 (right hand side). Heuristically, we think of $V_C$, for large enough $C$, as corresponding to this tangle. The loop around the singular value corresponds to the elementary braid (as illustrated on the left hand side). Extending $V_C$ over this loop by symplectic parallel transport, one sees already an important invariance result. Namely, $V_C$ is unchanged, up to an appropriate isotopy. This corresponds to the $(n-2,n)$–tangle being unchanged, up to isotopy, by composition with the braid.

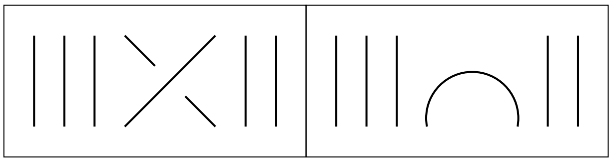

Figure 12: An elementary braid (left) and a basic "cap" tangle (right) which is unchanged by the action of that braid

**Definition 4.1.** Let $S_n$ be the space degree $n$ monomials with coefficients in the ring of 2 by 2 matrices in $\mathbb{C}[X]$ and such that the coefficient of $X^{n-1}$ is trace free. The determinant map $\chi$ restricted to $S_n$ always gives monomials of degree $2n$ with roots summing to zero, so can be thought of as a map to $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ (by identifying monomials with the configuration of their roots counting multiplicities). This is an example of the type of fibration just discussed.

The original definition of $S_n$ was as a nilpotent slice of the Lie algebra $\mathfrak{sl}_{2n}(\mathbb{C})$, specifically a local transverse slice to the orbit of the Adjoint action of $\mathrm{SL}_{2n}(\mathbb{C})$ at a nilpotent matrix with precisely 2 Jordan blocks of size $n$. This setting is described more explicitly as follows.

**Definition 4.2.** One defines $S_n$ to consist of the following $2n$ by $2n$ matrices with complex coefficients:

$$\begin{pmatrix} A_1 & I & & \\ \vdots & & \ddots & \\ \vdots & & & I \\ A_n & & & 0 \end{pmatrix}$$

where each $A_i$ is a 2 by 2 matrix (i.e. in $\mathfrak{gl}_2(\mathbb{C})$) and $A_1$ is trace free (in $\mathfrak{sl}_2(\mathbb{C})$). Then the map $\chi$ is defined to give the characteristic polynomial of the matrix.

The two definitions are related by the isomorphism:

$$\begin{pmatrix} A_1 & I & & \\ \vdots & & \ddots & \\ \vdots & & & I \\ A_n & & & 0 \end{pmatrix} \mapsto X^n - \sum_{i=1}^{n} X^{n-i} A_i$$

which commutes with the map $\chi$ on either side. In either setting we shall denote the fibre of $S_n$ over a configuration $P$ by $\mathcal{Y}_{n,P}$.

Alternatively, it has been shown by Manolescu [19] that, for $P \in \mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$, the fibre $\mathcal{Y}_{n,P}$ injects holomorphically into the Hilbert scheme $\mathrm{Hilb}^n(M_n(P))$. Here, $M_n(P)$ is the 2–dimensional (over $\mathbb{C}$) Milnor fibre associated to the $A_{2n}$ singularity. It is explicitly described as a smooth affine surface by the equation $u^2 + v^2 + P(z) = 0$ in $\mathbb{C}^3$. The symplectic parallel transport maps (defined appropriately on compact subsets containing the relevant Lagrangian submanifolds) are, in an appropriate sense, lifts of Dehn twists on $M_n(P)$. This reveals a close connection between the symplectic geometry of $S_n$ with the braid group $Br_{2n}$ which acts naturally by Dehn twists on the $A_{2n}$ Milnor fibre.

With an appropriate choice of Kähler metric [1] uses “rescaled” symplectic parallel transport maps. For this paper however, it is more convenient to use actual symplectic parallel transport. This has the consequence that we often need to perform some deformation in order to ensure that the symplectic parallel transport is well-defined on relevant compact subsets. Given any smooth path in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ we define the symplectic parallel transport maps on sufficiently large compact subsets of a given fibre by the deformation described in Lemma 2.19. Specifically Remark 2.21 covers this case.

The singular locus of $S_n$ corresponding to two coordinates in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ meeting at zero is precisely $\mathrm{MBL}(\chi)$ (see Section 2.2).

Furthermore, it is shown that $S_{n-1}$ is canonically isomorphic to a singular locus in $S_n$ over the points $(0, 0, \mu_1, \ldots, \mu_{2n})$ in a manner compatible with the fibrations. Local neighbourhoods have a particularly nice form.

**Lemma 4.3.** *(cf. Lemma 27 of [1])*

*Let $D$ be a disc in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ given by the coordinates*

$$(-\sqrt{\epsilon}, \sqrt{\epsilon}, \mu_3, \ldots, \mu_{2n})$$

*with $\epsilon$ small. Then there is a neighbourhood of $S_{n-1} \subset \chi^{-1}(D) \bigcap S_n$ on which the fibration $\chi$ has the local model given by a neighbourhood of $\mathcal{Y}_{n-1,(\mu_3,\ldots,\mu_{2n})} \times 0$ below:*

$$\begin{tikzcd}
\chi^{-1}(D) \arrow[rrr, "local\ isomorphism\ near\ S_{n-1}\subset\chi^{-1}(D)\bigcap S_n"] \arrow[d, "\chi"'] & & & \mathcal{Y}_{n-1,(\mu_3,\dots,\mu_{2n})} \times \mathbb{C}^3 \arrow[d, "a^2+b^2+c^2"] \\
D \arrow[rrr, "(-\sqrt{\epsilon},\sqrt{\epsilon},\mu_3,\dots,\mu_{2n})\mapsto\epsilon"'] & & & \mathbb{C}
\end{tikzcd}$$

In fact, one can drop the requirement that the two coordinates meet at zero (rather than an arbitrary point in $\mathbb{C}$) in the case that $n \geq 2$.

Let $L$ be a compact exact Lagrangian submanifold of $\mathcal{Y}_{n-1,(\mu_3,\dots,\mu_{2n})}$. Then the relative vanishing locus to $L$ (over simple enough paths into the singular value of the model fibration in the Lemma) gives an exact Lagrangian submanifold of a nearby regular fibre $\mathcal{Y}_{n,u(z)}$ of $S_n$ (cf. [1, Section 4.2]). It is diffeomorphic to $L \times S^2$ and well-defined up to Lagrangian isotopy, which, since $\pi_1(L \times S^2) = 0$, implies it is well-defined up to compactly supported Hamiltonian isotopy. Rezazadegan [8] has an approach to the non-compact case, but it is not necessary here.

By starting with $L = \{0\} = \mathcal{Y}_{0,0}$ and repeating this construction we can generate compactly supported Hamiltonian isotopy classes of Lagrangian submanifolds corresponding to isotopy classes of certain $(0, 2n)$–tangles.

**Definition 4.4.** Let $\gamma_i$ for $i = 1, 2, \dots, n$ be a sequence of vanishing paths in $\mathrm{Conf}^0_{2i}(\mathbb{C})$ from non-singular values $z_i$ to singular values $w_i$. Suppose also that, for each $i = 1, 2, \dots, n-1$ we have $X^2 z_i = w_{i+1}$. Then we can consider the composition of the paths $X^{2(n-i)}\gamma_i$ as a piecewise smooth path in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$. We call such a path an *iterated vanishing path.* Viewed in terms of configurations of points in $\mathbb{C}$, such a path describes a way of bringing together $2n$ points into pairs (fixed at 0).

Repeating the relative vanishing cycle construction along a short enough iterated vanishing path, starting with a single point inside $\{0\} \cong \mathcal{Y}_{0,0}$, allows one to construct a Lagrangian *iterated vanishing cycle.* Given a longer iterated vanishing path, this construction also works, though it becomes necessary to perform a deformation (described in Lemma 2.19 and Remark 2.21) to ensure that symplectic parallel transport of the vanishing loci is always well-defined.

It is shown in [1] that iterated vanishing cycles (up to isotopy) are independent of isotopy of the iterated vanishing paths defining them.

**Definition 4.5.** A *crossingless matching* of $P \in \mathrm{Conf}_{2n}(\mathbb{C})$ is an embedded set of n curves in $\mathbb{C}$, such that each coordinate of $P$ is an endpoint of precisely one curve. We denote the set of crossingless matchings on $P$ by $\mathfrak{M}^n_P$.

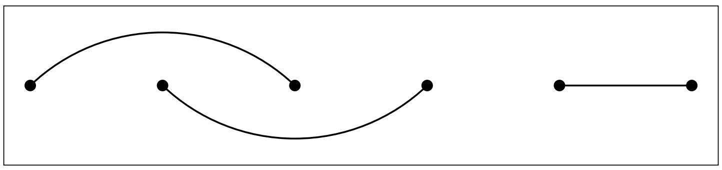

Figure 13: An example crossingless matching for $P \in \mathrm{Conf}_6(\mathbb{C})$.

It suffices to specify a crossingless matching $A \in \mathfrak{M}^n_P$ together with an ordering on the curves in order to determine an iterated vanishing cycle up to isotopy. To construct the iterated

vanishing cycle, one takes any $\gamma\colon [0,1] \longrightarrow \overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ whose coordinates in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ sweep out the curves of $A$ in $\mathbb{C}$, bringing together endpoints of the curves in the chosen order. Choices of such $\gamma$ are all isotopic to each other and these isotopies induce exact isotopies of the iterated vanishing cycles. In fact, changing the ordering also changes the iterated vanishing cycle [1] only by exact isotopy.

By [20] any link $\mathfrak{L}$ can be put into *braid position* by isotopy, i.e. a position of the form shown in Figure 14 for some braid $\beta$ on $n$ strands. This position can be thought of as splitting into two identical crossingless matchings (on some $P \in \mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$) and a braid on $2n$ strands consisting of $\beta$ on the left-hand $n$ strands and the trivial braid on the others as in the picture.

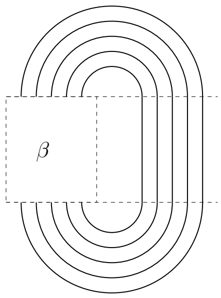


Figure 14: Braid position of a link, split as two identical crossingless matchings and a braid, trivial on the strands on one side.

Let $L \subset \mathcal{Y}_{n,P}$ be the Lagrangian corresponding to the crossingless matching and $L'$ be its image under the parallel transport corresponding to $\beta$. Then the *symplectic Khovanov homology* is defined as $\mathrm{KH}_{\mathrm{symp}}(\mathfrak{L}) := HF(L, L')$.

It follows from general results on Floer cohomology under symplectomorphisms and Hamiltonian isotopy, that $\mathrm{KH}_{\mathrm{symp}}$ is independent of braid isotopy. To prove that $\mathrm{KH}_{\mathrm{symp}}$ did not depend on the choice of braid representing a given link, Seidel and Smith show that it is invariant under the two *Markov moves*. These are changes to the braid which suffice to pass between any two braid positions of the same link.

Invariance under conjugation of the braid (*Markov I*) works straightforwardly by a trick involving isotopy of crossingless matchings.

The *Markov II* move (Figure 15) involves adding an extra strand to the braid $\beta$ and "twisting it in" by either a positive or a negative half twist. The proof of invariance under this move is more involved and makes use of a local model for the fibration of $S_n$ over a neighbourhood of a point in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ with three coordinates coinciding.

Another way in which parts of $S_{m-1}$ are locally nested in $S_m$ is as follows. Let $P \in \mathrm{Conf}^0_{2(\mathrm{m}\text{-}1)}(\mathbb{C})$ be a polynomial with pairwise distinct roots, one of which is 0. Then it turns out that the singular locus of $S_m$ over $X^2P$ is precisely $X^2\mathcal{Y}_{m-1,P}$. There is also a simple holomorphic model for neighbourhoods of this locus inside $S_m$. Namely one can describe it as a holomorphic associated bundle over $\mathcal{Y}_{m-1,P}$ in the following way.

**Lemma 4.6.** *(cf. Lemma 29 of [1])*

*Let* $XP \in \mathrm{Conf}^0_{2(\mathrm{m}\text{-}1)}(\mathbb{C})$ *be a polynomial with pairwise distinct roots one of which is* 0. *Let* $\mathbb{D}^2$ *be a small holomorphic bidisc in* $\overline{\mathrm{Conf}}^0_{2m}(\mathbb{C})$ *parametrised by*

$$(d, z) \mapsto (X^3 - Xd + z)P$$

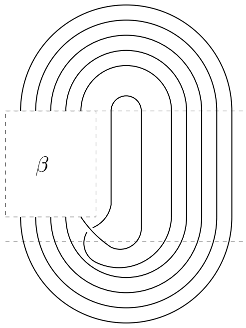


Figure 15: The braid position of Figure 14 after application of the Markov II move

*There is a holomorphic line bundle $\mathcal{F}$ over $\mathcal{Y}_{m-1,XP}$ such that the following is a local model for $\chi^{-1}(\mathbb{D}^2)$.*

$$\begin{tikzcd}
\chi^{-1}(\mathbb{D}^2) \arrow[rrr, "\textit{local isomorphism near } X^2\mathcal{Y}_{m-1,XP}\cap\chi^{-1}(0,0)"] \arrow[d, "\chi"'] & & & (\mathcal{F}\setminus 0)\times_{\mathbb{C}^*}\mathbb{C}^4 \arrow[d, "f"] \\
\mathbb{D}^2 \arrow[rrr] & & & \mathbb{C}^2
\end{tikzcd}$$

*The map $f$ is induced by the map $\mathbb{C}^4 \longrightarrow \mathbb{C}^2$ taking $(a,b,c,d)\mapsto(d,a^3-ad+bc)$. The $\mathbb{C}^*$–action on $\mathbb{C}^4$ is given by $(a,b,c,d)\mapsto(a,\zeta^{-2}b,\zeta^2c,d)$ for $\zeta\in\mathbb{C}^*$.*

I have stated a simple version of the Lemma. In fact Lemma 29 of [1] is more general in that it gives a local model near the fibre over a polynomial $P'\in\overline{\mathrm{Conf}}^0_{2m}(\mathbb{C})$ with root $\mu$ of multiplicity 3 and the remaining roots pairwise distinct. The difference is that $\mu$ is not required to be zero. This generality is necessary (though only in some cases).

The use of this local model is as follows. One considers the fibration over discs defined by a fixed value of $d\neq 0$. $\mathbb{C}^3\times\{d\}\longrightarrow\mathbb{C}$ is an exact Lefschetz fibration with two singular values. These each correspond to moving two of the three roots of $(X-\mu)^3-(X-\mu)d+z$ together in different ways. This makes $(\mathcal{F}\setminus 0)\times_{\mathbb{C}^*}\mathbb{C}^3$ an exact Morse Bott Lefschetz fibration over $\mathbb{C}$.

Let $K,K'$ in $\mathcal{Y}_{m-1,XP}$ be the iterated vanishing cycles constructed for some braid position of a link. One performs, with each of $K,K'$, the relative vanishing cycle construction in $(\mathcal{F}\setminus 0)\times_{\mathbb{C}^*}\mathbb{C}^3$ to create $L,L'$. This corresponds to adding a new unlinked component to the link. Then $L$ is carried around the other singularity by parallel transport to $\tau L$, corresponding to twisting the new component into the braid as in Figure 15. When this is done compatibly with splitting of the fibration from the Lemma one finds a bijection of the moduli spaces used to calculate $HF(K,K')$ and $HF(\tau L,L')$.

With an appropriate grading shift, this takes care of invariance under the Markov $II$ move. The important part to notice is that, by Lemma 4.6, the fibration splits in such a way as to separate the local behaviour of the three coordinates involved from the rest of the link. This is studied in more detail later.

## 4.2 Symplectic Khovanov homology for links in bridge position

Here we generalise the definition of $\mathrm{KH}_{\mathrm{symp}}$ such that it is defined directly using any bridge position of a link, not necessarily just braid positions. This will be useful in defining the maps

on $\mathrm{KH}_{\mathrm{symp}}$ corresponding to smooth closed cobordisms in $\mathbb{R}^4$ between links. These maps will be covered in the next section.

Any link in $\mathbb{R}^3 = \{(x, y, z)\}$ is isotopic to one in which the height function (mapping points on the link to the value of their $z$–coordinate) has only non-degenerate critical points. Furthermore we may require all local maxima of $z$ to occur where $z > 0$ and minima where $z < 0$. We will call such a position of a link *admissible*. Each half of a link in an admissible position (i.e. the parts above and below height $z = 0$) specifies a path for the iterated vanishing cycle construction and hence a Lagrangian submanifold of the same fibre $\mathcal{Y}_{n,P}$. One should note that these paths need not be embedded or disjoint from each other.

An isotopy of a link through admissible positions fixing the point $P \in \mathrm{Conf}^0_{2n}(\mathbb{C})$ causes Hamiltonian isotopies of this pair of Lagrangians. On Floer cohomology these give the identity up to canonical isomorphisms (from continuation maps). Although the Floer cohomology may arise from a different chain complex, before and after the isotopies, it is well defined up to canonical isomorphism.

Now we extend this definition to bridge diagrams of links and use it to prove that the Floer cohomology one gets for any two bridge diagrams of the same link is always isomorphic (though not with a canonical isomorphism unless one is comparing bridge positions coming from isotopic admissible positions).

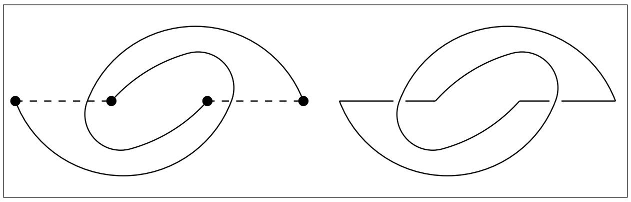

Figure 16: The Hopf link as a bridge diagram and a projection of the admissible position the diagram represents ($\alpha$–curves are drawn smooth, $\beta$–curves dashed)

**Definition 4.7.** An *$n$–bridge diagram* $(P, A, B)$ of a link consists of a set $P$ of $2n$ points in the plane together with a pair of crossingless matchings $A, B$ (sets of $n$ disjoint curves in the plane which join the points of $P$ in pairs) which intersect transversely away from $P$. We refer to the curves in these matchings as *$\alpha$– and $\beta$–curves* respectively and to the points of $P$ as the *vertices* of the diagram.

Any bridge diagram is easily turned into an admissible link by pulling the $\alpha$–curves up out of the plane in which the diagram is drawn and pushing the $\beta$–curves down. Such an admissible position will project back onto the bridge diagram by the orthogonal projection onto the plane. In fact any two such admissible positions are isotopic by an isotopy of admissible positions fixing $P$. Therefore we have a canonical choice of isomorphisms between the Floer homologies they induce.

**Definition 4.8.** Given an admissible position $\mathfrak{L}$ for a link or bridge diagram $(P, A, B)$ we define its *symplectic Khovanov homology* $\mathrm{KH}_{\mathrm{symp}}(\mathfrak{L})$ or $\mathrm{KH}_{\mathrm{symp}}(P, A, B)$ to be $HF(L_A, L_B)$. Here $L_A$, $L_B$ are constructed by the iterated vanishing cycle construction for a path specified by $\mathfrak{L}$, $(P, A, B)$ respectively.

From the above discussion, we see that this gives a well defined invariant of admissible positions, up to isotopy fixing the point $P \in \mathrm{Conf}^0_{2n}(\mathbb{C})$, and also of bridge diagrams up to isotopy fixing $P$.

A *regular isotopy* of bridge diagrams is an isotopy $P(t)$ of $P$ together with isotopies of $A$ and of $B$ through crossingless matchings on $P(t)$. The intermediate matchings need not intersect transversely.

A *passing move* on a crossingless matching is where we take a smooth loop $\gamma$ disjoint from the curves of the matching and enclosing precisely one of the curves and replace another of the curves with its connect sum with $\gamma$.

A *stabilisation move* on an $n$–bridge diagram is as follows. We mark a closed interval on an $\alpha$–curve $\gamma$ which is disjoint from the $\beta$–curves and from $P$. We then add its two endpoints to $P$, add the interval to $B$, replace $\gamma$ in $A$ with the two curves we get by removing the interior of the interval from $\gamma$. The resulting $(P, A, B)$ is an $(n+1)$–bridge diagram for the same link.

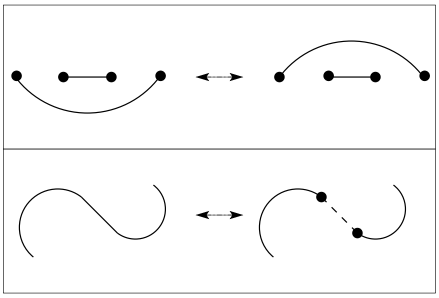

Figure 17: The *passing move* (above) and the *stabilisation move* (below)

Any bridge diagram $D$ naturally yields a crossing diagram $\mathrm{proj}(D)$ by projecting the construction in $\mathbb{R}^2 \times [-1, 1]$ to $\mathbb{R}^2$. There are, however, many different bridge diagrams that yield the same crossing diagram by this method. All such diagrams are related by a finite sequence of stabilisation moves (and destabilisation moves) and regular isotopy fixing the projection.

**Lemma 4.9.** *Finite sequences of the three moves described above applied to any bridge diagram $D$ suffice to perform all Reidemeister moves on* $\mathrm{proj}(D)$.

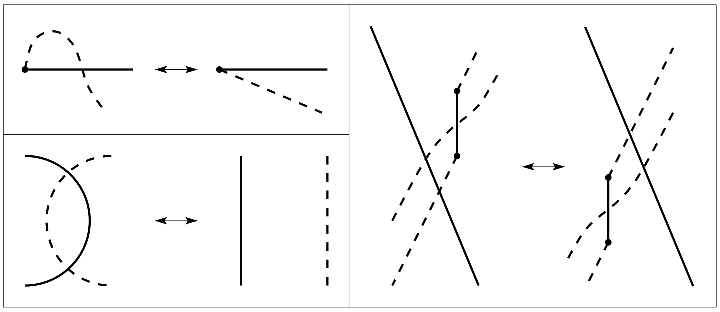

Figure 18: The Reidemeister moves (one way up) after some destabilisation moves are the projections of the above moves

*Proof.* By repeated destabilisation, the neighbourhood in $\mathbb{R}^2$ in which one performs a Reidemeister move can be made to be one of those in Figure 18 (up to a reflection and/or swapping the $\alpha$– and $\beta$–curves). The illustration makes it clear that isotopy suffices to perform the Reidemeister I and II moves and the passing move to perform the Reidemeister III move. ☐

**Lemma 4.10.** *Finite sequences of these three moves suffice to go between any two bridge diagrams of the same link.*

*Proof.* Given $D_1$, $D_2$, bridge diagrams of the same link, there is a sequence of Reidemeister moves from proj($D_1$) to proj($D_2$). By the previous Lemma, there is a finite sequence of moves taking $D_1$ to some $D_3$ with proj($D_3$) = proj($D_2$). Now a finite sequence of stabilisation and destabilisation moves relates $D_3$ and $D_2$. $\square$

Isotopy of $P$ to $Q$ along a path in $\mathrm{Conf}^0_{2n}(\mathbb{C})$ gives an exact symplectomorphism from (any compact subset of) $\mathcal{Y}_{n,P}$ to $\mathcal{Y}_{n,Q}$ by symplectic parallel transport. Given an isotopy of admissible links which induces this same isotopy $P$ to $Q$ on its intersection with the plane $z = 0$, this symplectomorphism carries the Lagrangians corresponding to the first admissible link to those for the second (up to Hamiltonian isotopy). Hence, it gives an isomorphism of symplectic Khovanov homology. Using this, one sees that regular isotopy of bridge diagrams and also the passing move do not change the isomorphism class of symplectic Khovanov homology.

**Lemma 4.11.** *Stabilisation of a bridge diagram gives a bridge diagram with isomorphic symplectic Khovanov homology.*

*Proof.* Modified by an isotopy of admissible links, the Markov $II^+$ move is just the stabilisation move (see Figure 19). The proof of invariance under the Markov $II^+$ move in [1] can be carried out in exactly the same manner in this setting since there is no part of it that requires the admissible links defining the Lagrangians to be of the particular form used in that paper.

The only difference is technical. The localisation to coordinates of the form in Lemma 4.6 must be carried out differently if we insist on using the construction of Lemma 2.19 instead of rescaled symplectic parallel transport in the iterated vanishing cycle construction. Rather than explain, Lemma 5.19 is a strictly stronger version of the above result. $\square$

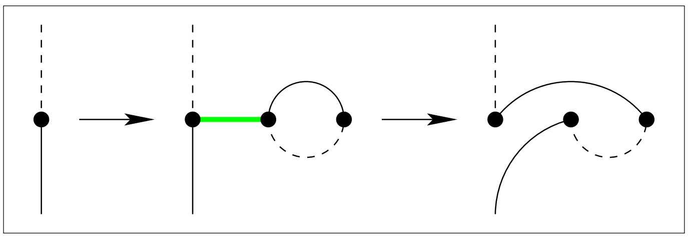

Figure 19: An illustration of the Markov $II^+$ move as seen in a bridge diagram. In the first step a new link component is created, then in the second step 'twisted in' by a positive twist along the thicker line.

In conclusion it has been shown in this section that:

**Theorem 4.12.** *The isomorphism class of* $\mathrm{KH}_{\mathrm{symp}}$ *of a bridge diagram as a relatively graded group is an invariant of link isotopy.*

**Remark 4.13.** It is worth noting here that the Markov $II^-$ move can equally well be used in the above. Namely the crossingless matchings one gets by replacing the positive twist in Figure 19 by a negative twist differ by an isotopy. In the setting of [1] $HF(L_A, L_B)$ carries an absolute grading which the Markov $II^\pm$ moves change by different, but fixed, amounts. An overall grading shift from the writhe of the link counteracts this to give an absolute grading on

$\mathrm{KH_{symp}}$. To define the same absolute grading for links in bridge position will take a little more care as these observations show that the absolute grading on $HF$ should not be preserved by all isotopies of admissible links. In fact a correction needs to be made for the passing moves to get the absolute grading on $\mathrm{KH_{symp}}$ in this setting.

**Remark 4.14.** By stretching all the local maxima in the upper half of an admissible link upwards one can decompose it as a trivial crossingless matching and a braid. Now considering the braid group action by diffeomorphisms on the plane containing this trivial matching, one finds that the upper tangle is isotopic to a tangle whose projection to the $(x, y)$–plane has no crossings. Doing the same with the lower half, the link can be taken by isotopy of the two halves to one which projects to a bridge diagram. Hence Theorem 4.12 also holds for $\mathrm{KH_{symp}}$ of any admissible link.

# 5 Maps on symplectic Khovanov homology from smooth cobordisms

In the setting of Khovanov homology one has maps between the chain complexes corresponding to elementary saddle cobordisms and creation/annihilation ('cap'/'cup') cobordisms as portrayed in a given crossing diagram. In this section we construct analogous maps on $\mathrm{KH_{symp}}$ and prove that, up to an overall sign ambiguity, they compose to give maps depending only on the isotopy class of the composite cobordism. Unless otherwise indicated, isotopies of a cobordism must fix the links at either end of the cobordism, since we generally want the symplectic Khovanov homology of the domain and range to be well-defined up to canonical isomorphism.

In the case of cobordisms which are themselves isotopies of links through admissible positions there is little difficulty. Namely, corresponding to isotopies of admissible links which fix the intersection with the plane $z = 0$ we have continuation maps between Floer cohomology groups. Suppose we have an isotopy of admissible positions which moves this intersection along a path $\gamma\colon \mathbb{R} \longrightarrow \mathrm{Conf}^0_{2n}(\mathbb{C})$ which is constant near $\pm\infty$. Then we can construct an admissible map

$$u_\gamma : \mathbb{R} \times [0, 1] \longrightarrow \mathbb{R} \longrightarrow \mathrm{Conf}^0_{2n}(\mathbb{C})$$

This defines an exact MBL–fibration (cf. Section 2.2) with two ends. Extending Lagrangians along the boundaries of this fibration allows us to calculate a relative invariant mapping between the symplectic Khovanov homologies of the admissible positions at either end of the isotopy.

In fact this map on $\mathrm{KH_{symp}}$ is the same as the map induced by the symplectic parallel transport along $\gamma$, but it is handy to have it given in terms of a symplectic fibration.

**Remark 5.1.** Deforming the admissible map $u_\gamma$ gives, by Lemma 2.38, the following result. Isotopies which are isotopic through isotopies of admissible positions of links (fixing the links at either end) give the same map on symplectic Khovanov homology. One should note however, that there exist isotopies from a link to itself which do not give the identity map on Khovanov homology [7, Theorem 1].

## 5.1 Saddle cobordisms

In this section I construct maps which correspond to certain saddle cobordisms. I shall begin by outlining how saddle cobordisms naturally arise from admissible maps into $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$.

Let $\mathfrak{L}$ be a link in an admissible position which intersects the $z = 0$–plane in the configuration $P$. Up to some choice, this defines (and is defined by) iterated vanishing cycles $\gamma_A, \gamma_B : [0,1] \longrightarrow \overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$. We shall assume both paths end in a segment of constant path at $P$.

For the moment, we shall restrict attention to link cobordisms supported close to $\{z = 0\}$. The parts of the trivial cobordism from $\mathfrak{L}$ to itself away from $\{z = 0\}$ can be realised as the following sets:

$$\{(t, z, \lambda) \in [-1, 1] \times [1, 2] \times \mathbb{C} \ : \ \lambda \text{ is a root of } \gamma_A(z-1)\}$$

$$\{(t, z, \lambda) \in [-1, 1] \times [-2, -1] \times \mathbb{C} \ : \ \lambda \text{ is a root of } \gamma_B(1-z)\}$$

I will now exhibit some link cobordisms by joining these surfaces up with a surface in the missing region $[-1, 1] \times [-1, 1] \times \mathbb{C}$.

Let $u : \overline{\mathbb{D}} \longrightarrow \overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ be an admissible map (see Definition 2.17) with two boundary marked points $\pm 1$, such that $u(-1) = P$. To make the cobordism we construct be smooth, we also require $u$ to be a constant map near each of the boundary marked points. Now choose your favourite smooth map $[-1, 1] \times [-1, 1] \longrightarrow \overline{\mathbb{D}}$ which is a diffeomorphism away from the boundaries and maps $\{\pm 1\} \times [-1, 1]$ to $\pm 1$ respectively. We shall call the composition of $u$ with this map $\tilde{u}$.

The following set defines a braid cobordism from the trivial braid to some other braid fixing the configuration $P$ at the ends:

$$\{(t, z, \lambda) \in [-1, 1] \times [-1, 1] \times \mathbb{C} \ : \ \lambda \text{ is a root of } \tilde{u}(t, z)\}$$

Hence, it can be inserted in the above construction to give a link cobordism, starting at $\mathfrak{L}$, which is supported near $\{z = 0\}$.

Now I shall describe how to construct an admissible map corresponding to certain elementary saddle cobordisms starting at the link $\mathfrak{L}$. As input to the construction, I take an embedded path $\delta : [0, 1] \longrightarrow \mathbb{C}$ ending at roots of $P$ and otherwise disjoint from them. The link cobordism I construct will be the elementary braid cobordism which adds a single negative half-twist along the curve $\delta$. For reasons of orientation of holomorphic embeddings in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$, the sign of the half-twist one adds along $\delta$ will be necessarily negative. However, this problem is actually an artefact of the viewpoint of braid cobordisms. There is no sign associated to elementary saddle cobordisms of links.

By fattening the curve $\delta$ one can define a smooth map $v : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ which:

- maps $\pm 1$ to the ends of $\delta$,
- otherwise misses all the roots of $P$,
- and is a holomorphic embedding near $0 \in \overline{\mathbb{D}}$

The space of choices of such a smooth map is contractible.

Let $\tilde{P}$ be the product of $X - r$ over all roots $r$ of $P$ which are not ends of $\delta$. Then we define the admissible map $u_\delta$ to be the following composite:

$$\begin{array}{ccccc}
\overline{\mathbb{D}} & \longrightarrow & \overline{\mathrm{Conf}}^0_2(\overline{\mathbb{D}}) & \longrightarrow & \overline{\mathrm{Conf}}^0_{2n}(\mathbb{C}) \\
z & \longmapsto & X^2 + z & & \\
 & & (X-a)(X-b) & \longmapsto & (X - v(a))(X - v(b))\tilde{P}
\end{array}$$

One can check that this is admissible. We fix an input boundary marked point at $-1$ and output boundary marked point at 1. In particular $u(-1)$ is the configuration $P$. In order to have this admissible map well-defined up to isotopy fixing the boundary marked points, I shall generally require that the $v$ is constant on the segments of $\partial\overline{\mathbb{D}}$ between 1 and $i$ and between -1 and $-i$. This causes the output boundary marked point to also be $P$ and the path in $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$ given by $u$ applied to the lower half of $\partial\overline{\mathbb{D}}$ to be contractible through $\mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$. Hence, the lower half (where $z < 0$) of the link one has at the output end of the cobordism is the same as the lower half at the input. This is a convenient simplification when one wishes to express these cobordisms in terms of bridge diagrams.

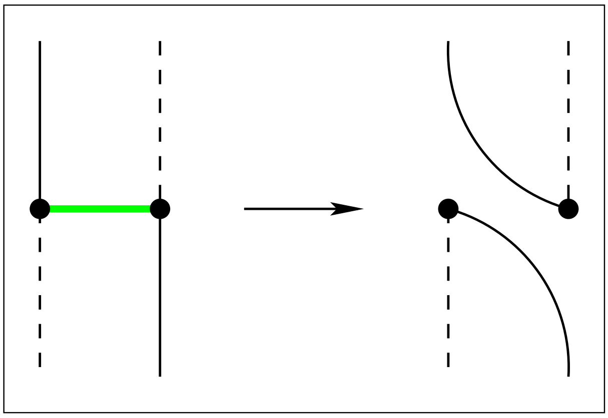

Figure 20: An elementary saddle cobordism specified by a curve in a bridge diagram. When projected to a crossing diagram, this example is also the usual elementary saddle cobordism between resolutions of a crossing

In a bridge diagram $(P, A, B)$ for $\mathfrak{L}$ I shall draw the curve $\delta$ in grey to indicate a cobordism about to be performed. Figure 20 gives an example of this with an arrow pointing to the bridge diagram for the output link. The simplification mentioned above allows us to fix $P$ and apply the entire half-twist to the curves from $A$, without changing $B$. The cobordism illustrated in Figure 20 is an elementary saddle cobordism of the more familiar kind under projection of the bridge diagram to a crossing diagram. From now on we shall view cobordisms "from above" in this manner.

It is now a simple matter to define the map $f_\delta$ that a cobordism specified by a curve $\delta$ induces on symplectic Khovanov homology.

**Definition 5.2.** Given a bridge diagram $(P, A, B)$ together with a curve $\delta$, as above, specifying a cobordism to the bridge diagram $(P', A', B^prime)$ we define the map

$$f_\delta : \mathrm{KH}_{\mathrm{symp}}(P, A, B) \longrightarrow \mathrm{KH}_{\mathrm{symp}}(P, A, B)$$

to be the relative invariant induced by the admissible map $u_\delta$ and the pair of Lagrangians used to define $\mathrm{KH}_{\mathrm{symp}}(P, A, B)$. This relative invariant is well defined by Lemma 2.38.

**Remark 5.3.** The relative invariant of this definition is well defined up to composition with the canonical isomorphisms of $\mathrm{KH}_{\mathrm{symp}}$ of the domain and range. To show this one simply applies Lemma 2.38. It is independent of isotopy of $\delta$ and of changing of $(P, A, B)$ to any other bridge diagram representing the same admissible link position (i.e. isotopy fixing $P$ and the passing move).

Composing two admissible maps given by elementary saddle cobordisms (by gluing striplike ends of the admissible maps) gives the composite of the maps on symplectic Khovanov homology. The two singular values for the composite admissible map can be moved past each other by an isotopy of admissible maps (see Figure 5. Decomposing the result into two admissible maps shows that the map on symplectic Khovanov homology can be expressed in different ways as a composite. This is a simple version of Lemma 2.26. It will be vital later to know which composites of saddle cobordisms give the same maps in this way.

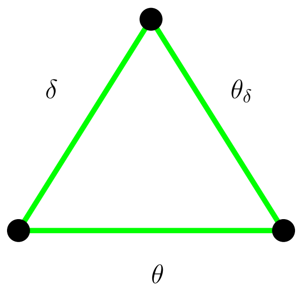


Figure 21: A simple example of curves $\delta$, $\theta$ and $\theta_\delta$ in the plane satisfying the conditions of Lemma 5.4.

**Lemma 5.4.** *Let $\delta$, $\theta$ be embedded curves in $\mathbb{C} \setminus P$, each joining two points of $P \in \mathrm{Conf}^0_{2n}(\mathbb{C})$. Write $\theta_\delta$ for the result of a positive half twist along $\delta$ applied to $\theta$. Then the maps these curves specify on any $\mathrm{KH}_{\mathrm{symp}}(P, A, B)$ satisfy the following relation:*

$$f_\theta \circ f_\delta = f_\delta \circ f_{\theta_\delta}$$

*Proof.* One glues the pairs of admissible maps for each composite together and observes that the resulting admissible maps are isotopic by an isotopy which moves the singular values past each other. $\square$

This is a minor abuse of notation. For example, the maps $f_\delta$ have different domain and range on either side of the equation. However, the composites map between the same (up to canonical isomorphism) cohomology groups.

**Corollary 5.5.** *Suppose $\delta$ and $\theta$ do not intersect in $\mathbb{C}$. Then the cobordisms specified by $\delta$ and $\theta$ can be performed in either order with the same resulting map on $\mathrm{KH}_{\mathrm{symp}}(P, A, B)$.*

## 5.2 A semi-canonical splitting of symplectic Khovanov homology

In this section I generalise the splitting of Lemma 4.3 in a way which allows one to split symplectic Khovanov homology of a link consisting of multiple unlinked components as a product. This is done in the case of 2 components, where one is an unknot, in Lemma 41 of [1]. We then strengthen the result by showing the splitting is well-defined up to canonical isomorphism on one factor and furthermore that one can split certain maps $f_\gamma$ induced by saddle cobordisms in a similar manner.

In the following we will use both descriptions of the fibrations $S_k$ given in Definitions 4.1 and 4.2 and freely switch between them. It should be clear from context which is meant.

We start the construction by observing that there is a natural choice of copy of $S_k$ in $S_{k+l}$, namely $X^l S_k$. Similarly $X^{2l}\overline{\mathrm{Conf}}^0_{2k}(\mathbb{C}) \subset \overline{\mathrm{Conf}}^0_{2l}(\mathbb{C})$ and these two inclusions are compatible with the fibrations. That is, for $y$ in $S_k$ we have $X^{2l}\chi(y) = \chi(X^l y)$.

Choose any polynomial $P = \prod_i (X-\mu_i) \in \mathrm{Conf}^0_{2k}(\mathbb{C}^*)$. Let $\xi$ be the linear subspace of $\mathbb{C}^{2k+1}$ given by the equation $2lz_0 + \sum_{i=1}^{2k} z_i = 0$. There is a neighbourhood of $X^l P \in \overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$ which is biholomorphic to a neighbourhood of $(X^{2l}, 0, P)$ in $\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) \times \mathbb{C} \times \overline{\mathrm{Conf}}^0_{2k}(\mathbb{C})$ and of $(X^{2l}, (0, \mu_1, \ldots, \mu_k))$ in $\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) \times \xi$. Explicitly the biholomorphisms are described (in the opposite order) by

$$\begin{aligned}
\left(\prod_{i=1}^{2l}(X-a_i), (z_0, z_1, \ldots, z_{2k})\right) &\mapsto \left(\prod_{i=1}^{2l}(X-a_i), z_0, \prod_{i=1}^{2k}(X - z_i - \frac{lz_0}{k})\right) \\
&\mapsto \prod_{i=1}^{2l}(X - a_i - z_0)\prod_{i=1}^{2k}(X - z_i)
\end{aligned}$$

We will now show that, for $y \in \mathcal{Y}_{k,P}$ we can locally model any transverse slice to $X^l y \in S_{k+l}$ on $S_l \times \xi$ compatibly with the description of a neighbourhood of $X^l P$ as $\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) \times \xi$. In fact these models can be chosen in a consistent manner for all $y \in \mathcal{Y}_{k,P}$ simultaneously:

**Proposition 5.6.** *(cf. Lemma 4.3)*
*There is a tubular neighbourhood of $X^l\mathcal{Y}_{k,P}$ in $S_{k+l}$ which is biholomorphic to a neighbourhood of $\mathcal{Y}_{k,P}$ in $\mathcal{Y}_{k,P} \times S_l \times \xi$. Furthermore the following diagram of holomorphic maps commutes:*

$$\begin{tikzcd}
S_{k+l} \arrow[d, "\chi"'] & \mathcal{Y}_{k,P} \times S_l \times \xi \arrow[l, "\text{local} \cong \text{ defined near } X^l\mathcal{Y}_{k,P}"'] \arrow[d, "\chi\times\mathrm{id}_\xi"] \\
\overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C}) & \overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) \times \xi \arrow[l, "\text{local} \cong \text{ defined near } X^{2l}P"']
\end{tikzcd}$$

We shall in general actually use the following simpler version (restricting to $0 \in \xi$) which suffices to describe the behaviour of links with multiple unlinked components.

**Corollary 5.7.** *Given any $P \in \mathrm{Conf}^0_{2k}(\mathbb{C}^*)$ we have the following local holomorphic model for $S_{k+l}$ defined:*

- *on a neighbourhood of $X^l\mathcal{Y}_{k,P}$ in $\chi^{-1}(P\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C})) \subset S_{k+l}$*
- *over a neighbourhood of $PX^{2l}$ in $P\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) \subset \overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$*

$$\begin{tikzcd}
\chi^{-1}(P\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C})) \arrow[d, "\chi"'] & \mathcal{Y}_{k,P} \times S_l \arrow[l, "\text{local} \cong \text{ defined near } X^l\mathcal{Y}_{k,P}"'] \arrow[d, "\chi"] \\
P\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) & \overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) \arrow[l, "\text{local} \cong \text{ defined near } 0"']
\end{tikzcd}$$

*The map of bases takes $Q$ to $PQ$. The map of total spaces restricts to $\mathcal{Y}_{k,P} \times \{X^l\}$ as multiplication $(y, X^l) \mapsto yX^l$.*

The proof of Proposition 5.6 comes from generalising Lemmas 24 to 27 of [1] and will be presented in an analogous manner for ease of comparison. The slices $S_n$ are thought of here as spaces of $2n \times 2n$ matrices.

**Lemma 5.8.** *For $y \in S_{k+l}$, projection to the first $2l$ coordinates of $\mathbb{C}^{2(k+l)}$ restricts to an injection*

$$\ker(y^l) \longrightarrow \mathbb{C}^{2l}$$

*Proof.* For $y \in S_{k+l}$ we observe that $y^l$ is of the form

$$\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}$$

where $A$ is a $2l \times 2k$ matrix and $B$ is a $2l \times 2l$ matrix. Hence, if $(v_1, \ldots, v_{2(k+l)}) \in \ker(y^l)$, then we have:

$$-A \begin{pmatrix} v_1 \\ \vdots \\ v_{2l} \end{pmatrix} = \begin{pmatrix} v_{2l+1} \\ \vdots \\ v_{2(k+l)} \end{pmatrix}$$

Hence the first $2l$ coordinates determine the rest. $\square$

**Lemma 5.9.** *The subspace of $y \in S_{k+l}$ such that $\ker(y^l)$ is $2l$–dimensional can be canonically identified with $S_k$. The identification is compatible with the adjoint quotient map and the inclusion $\overline{\mathrm{Conf}}^0_{2k}(\mathbb{C}) \subset \overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$ previously mentioned.*

**Remark 5.10.** This Lemma describes the inclusion, written earlier as $X^l S_k \subset S_{k+l}$. In particular it shows that for $P \in \mathrm{Conf}^0_{2k}(\mathbb{C}^*)$ the set $X^l S_k$ can be described as the transverse intersection of $S_{k+l} \subset \mathfrak{sl}_{k+l}(\mathbb{C})$ with an orbit of the adjoint action of $SL_{k+l}(\mathbb{C})$. This orbit is furthermore given by the property that its elements have 2 Jordan blocks, each of size $l$, for the eigenvalue 0. Hence transverse slices to elements of the orbit should contain $(l,l)$*–type nilpotent slices*, i.e. copies of $S_l$. The proof of Proposition 5.6 will be based upon this observation.

*Proof.* Let $y \in S_{k+l}$ have kernel of dimension $2l$ over $\mathbb{C}$. We consider the Jordan blocks for $y^l$ for eigenvalue 0. Lemma 5.8 in the case $l = 1$ says that there are at most two. Hence there must be precisely 2 and they must both have size at least $l$. This means the injection of Lemma 5.8 is an isomorphism for such $y$.

We will write vectors $v \in (\mathbb{C}^2)^{(k+l)}$ with coordinates $v_1, \ldots, v_{(k+l)} \in \mathbb{C}^2$. Now, observe that $y^i(v)_1 = M + v_{i+1}$ where $M$ depends only on $v_1, \ldots, v_i$. We prove, by induction on $i$, that $A_{k+1-i} = 0$.

In the case $i = 1$, we have $y(v)_k = A_k v_1$. In order for there to be a $v \in \ker(y)$ for any choice of $v_1$, we must have $A_k = 0$. For $i > 1$ (by applying $A_{k-i+2}, \ldots, A_k = 0$), we have:

$$\begin{aligned} y^i(v)_{k+1-i} &= y\left(y^{i-1}(v)\right)_{k+1-i} \\ &= A_{k+1-i}(y^{i-1}(v)_1) + y^{i-1}(v)_{k+2-i} \\ &= A_{k+1-i}(M + v_i) + A_{k+2-i}(y^{i-2}(v)_1) + y^{i-2}(v)_{k+3-i} \\ &\vdots \\ &= A_{k+1-i}(M + v_i) \end{aligned}$$

Here again $M$ depends only on $v_1, \dots, v_{i-1}$, so in order for there to be a $v \in \ker(y^i)$ for any choice of $v_1, \dots, v_i$ we must have $A_{k+1-i} = 0$.

We have just shown that the space of $y \in S_{k+l}$ with $\dim \ker(y^l) = 2l$ is contained in the space of those $y$ with $A_{k+1}, \dots, A_{k+l} = 0$. In fact one can check they are the equal. This space is canonically isomorphic to $S_k$ by the map

$$f : \begin{pmatrix} A_1 & I & & \\ \vdots & & \ddots & \\ \vdots & & & I \\ A_k & & & \end{pmatrix} \mapsto \begin{pmatrix} A_1 & I & & & \\ \vdots & & \ddots & & \\ A_k & & & \ddots & \\ 0 & & & & \ddots \\ \vdots & & & & & I \\ 0 & & & & & \end{pmatrix} \tag{1}$$

This has the property that $X^{2l}\chi(y) = \chi(f(y))$, so it commmutes with the inclusion

$$X^l \overline{\mathrm{Conf}}^0_{2k}(\mathbb{C}) \subset \overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$$

Also note that this map takes $y$ to $X^l y$ in the polynomial presentation of $S_{k+l}$. $\square$

With these results we now prove Proposition 5.6.

*Proof.* First we choose for every $y \in X^l \mathcal{Y}_{k,P}$ a local affine linear transverse slice varying holomorphically with $y$. It suffices (Lemma 5 (i) of [1]) to pick these slices transverse to the adjoint orbit of $y$ within $S_{k+l}$, i.e. from a complement to $T_y X^l \mathcal{Y}_{k,P}$ in $T_y S_{k+l}$. For example one can take the following. First choose a complement $W_y$ to $T_y \mathcal{Y}_{k,P}$ in $T_y S_k$ varying holomorphically with y. This splitting problem has a positive solution because $\mathcal{Y}_{k,P}$ is an affine subvariety of $S_k$. Then take the direct sum of $V \oplus X^l W$ with the orthogonal complement to $X^l S_k$ as a vector subspace of $S_{k+l}$.

Another way to choose a transverse slice to $X^l y$ is (by Lemma 8 of [1]) to decompose $\mathfrak{sl}_{2(k+l)}(\mathbb{C})$ into eigenspaces of the semisimple part $(X^l y)_s$ of $X^l y$. All the eigenspaces are 1–dimensional with the exception of the 0–eigenspace which is canonically identified with $\mathbb{C}^{2l}$ by Lemma 5.8. This gives a holomorphically varying local transverse slice.

$$\begin{array}{c} X^l y_s + S_l + \mathfrak{z} \\ \Big\downarrow{\scriptstyle \chi \times f} \\ \overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) \times \xi \end{array}$$

where $f$ is a local biholomorphism $\mathfrak{z} \longrightarrow \mathbb{C} \times \mathrm{Conf}^0_{2k}(\mathbb{C}^*)$.

Any two local transverse slices at a point are locally isomorphic by an isomorphism that moves points only inside their adjoint orbits (Lemma 5 (iii) of [1]) and hence does not affect $\chi$. This gives us the required result so long as this isomorphism can be chosen to vary holomorphically with $y$. In fact the isomorphism depends only on a choice of local submanifold $K_y \subset SL_{k+l}(\mathbb{C})$ near the identity element $e$ with tangent space at $e$ complementary to that of the adjoint orbit of $y_s$. The choice $K_y = \exp[\mathfrak{sl}_{k+l}(\mathbb{C}), y_s]$ suffices (cf. proof of Lemma 27 of [1]). $\square$

I shall now demonstrate how one can split symplectic Khovanov homology using appropriate *localisation* arguments. We begin with a useful, if obvious, Lemma related to shrinking components of a link.

**Lemma 5.11.** *Let $D$ be a disc centred at $0 \in \mathbb{C}$ and let*

$$P = P_1 P_0 \in \overline{\mathrm{Conf}}^0_{2k}(\mathbb{C} \setminus D) \times \mathrm{Conf}^0_{2l}(D) \subset \overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$$

*Suppose we have a bridge diagram $(P, A, B)$ which splits as the union of bridge diagrams $(P_0, A_0, B_0)$ supported on $D$ and $(P_1, A_1, B_1)$ supported outside $D$. See Figure 22 for an example. Then there is a canonical isotopy of bridge diagrams starting at this bridge diagram and ending at one which splits as the union of $(P_1, A_1, B_1)$ with $(P_0, A_0, B_0)$, the latter scaled in $\mathbb{C}$ by any $\lambda \in (0, 1)$. Hence, rescaling defines canonical isomorphisms on* $\mathrm{KH}_{\mathrm{symp}}$ *(or canonical chain homotopy classes of chain maps at the cochain level).*

*One can scale down any admissible map of a surface with boundary marked points into*

$$P_1 \,\mathrm{Conf}^0_{2l}(D) \subset \overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$$

*to obtain another admissible map, such that the relative invariant of the two surfaces are related by composing on either side by the canonical isomorphisms mentioned above.*

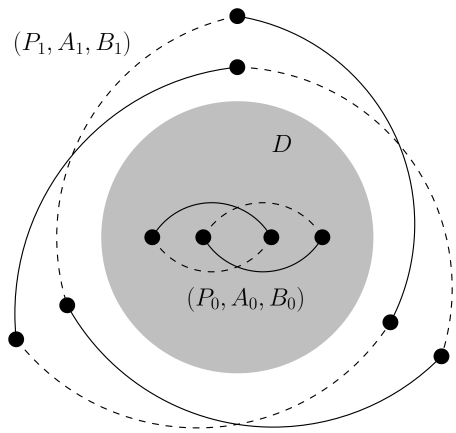


Figure 22: A diagram of a link with multiple unlinked parts arranged as specified in Lemma 5.11 and Theorem 5.12

*Proof.* The first part is obvious. For the second part one just needs to check that the $\mathbb{C}^*$–action on $\overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$, rescaling all roots by a complex number, preserves the property of a surface being admissible. The result comes from the isotopy of admissible curves one gets by continuously performing the scaling and composing on either end by the isotopies from the first part. ☐

**Theorem 5.12.** *Let $D$ be a disc centred at $0 \in \mathbb{C}$. Suppose we have a bridge diagram $(P, A, B)$ which splits as the union of bridge diagrams $(P_0, A_0, B_0)$ supported on $D$ and $(P_1, A_1, B_1)$ supported outside $D$.*

*Then* $\mathrm{KH}_{\mathrm{symp}}(P, A, B)$ *is isomorphic to the Künneth product of* $\mathrm{KH}_{\mathrm{symp}}(P_0, A_0, B_0)$ *and* $\mathrm{KH}_{\mathrm{symp}}(P_1, A_1, B_1)$*. Furthermore, this isomorphism is canonical up to composition with automorphisms of the* $\mathrm{KH}_{\mathrm{symp}}(P_0, A_0, B_0)$*–factor.*

**Remark 5.13.** We only apply this theorem in cases where one factor has no torsion (e.g. it is an unlinked union of unknots), so the Künneth product on cohomology is an honest tensor product.

We shall denote by $\Psi$ the local isomorphism given in Corollary 5.7

$$\mathcal{Y}_{k,P_1} \times S_l \xrightarrow{\Psi} \chi^{-1}(P_1\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}))$$

The idea is now to *localise* the calculation of Floer cohomology to the model region $\mathcal{Y}_{k,P} \times S_l$ with Lagrangians and exact symplectic structures which respect the splitting. In very specific cases (where $(P_0, A_0, B_0)$ is the simplest bridge diagram for a single unknot, or also a union of such bridge diagrams), one can use the same argument as for Lemma 41 of [1].

The proof I give below necessarily uses a more complicated argument to control the position of the Lagrangian iterated vanishing cycles to ensure they are contained in the region where $\Psi$ is defined. Otherwise it follows essentially the same argument. I begin by outlining the steps of the proof.

We start with the following information:

- Configurations $P_1 \in \overline{\mathrm{Conf}}^0_{2k}(\mathbb{C})$ and $P_0 \in \overline{\mathrm{Conf}}^0_{2l}(\mathbb{C})$
- Iterated vanishing paths $\gamma_{A_1}, \gamma_{B_1}$ in $\overline{\mathrm{Conf}}^0_{2k}(\mathbb{C})$ representing the crossingless matching $(P_1, A_1, B_1)$
- Iterated vanishing paths $\gamma_{A_0}, \gamma_{B_0}$ in $\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C})$ representing the crossingless matching $(P_0, A_0, B_0)$
- Any choice of exhausting plurisubharmonic functions $\rho_l$, $\rho_k$ and $\rho_{k+l}$ on $S_l$, $S_k$, $S_{k+l}$ respectively, such that $\rho_k$ is the restriction of $\rho_{k+l}$ to $X^l S_k \subset S_{k+l}$
- Exact Kähler structures $\Omega_k = d\Theta_k, \Omega_l = d\Theta_l, \Omega_{k+l} = d\Theta_{k+l}$ defined by the above plurisubharmonic functions
- Our favourite choices of Lagrangian iterated vanishing cycles $K_A, K_B \subset \mathcal{Y}_{k,P_1}$ over paths $\gamma_{A_1}, \gamma_{B_1}$ respectively

Abusing notation, I shall refer to the restrictions of $\rho_k, \Omega_k, \Theta_k$ to $\mathcal{Y}_{k,P_1}$ by the same names where it is clear from context which is meant. Similarly, I refer to the restrictions of $\rho_{k+l}, \Omega_{k+l}, \Theta_{k+l}$ to $\chi^{-1}(P_1\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}))$ by the same names.

The vanishing paths $\gamma_{A_1}$ and $P_1\gamma_{A_0}$ together give an iterated vanishing path $\gamma_A$ in $\overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$. We refer to this as the *composite*. One defines $\gamma_B$ similarly. The symplectic Khovanov homology cochain complex $\mathrm{CKH}_{\mathrm{symp}}(P, A, B)$ may be chosen to be the Floer cochain complex for Lagrangian submanifolds $L_A, L_B \subset \mathcal{Y}_{k,P}$ defined as iterated vanishing cycles along the vanishing paths $\gamma_A, \gamma_B$ respectively.

It will be necessary to explicitly specify the deformations performed to control symplectic parallel transport over the paths $\gamma_{A_0}$ and $\gamma_{B_0}$. Smoothly varying these choices will then give continuation maps between Floer cohomology groups. We may suppose (by isotopy of vanishing paths) that both $\gamma_{A_0}$ and $\gamma_{B_0}$ are the composite of smooth paths in $\mathrm{Conf}^0_{2l}(\mathbb{C})$ with the vanishing path constructed as a piecewise smooth composite of linear paths $X^{2(n-1)}\prod_{i=n}^{l}(X^2-\epsilon_i)$ to $X^{2n}\prod_{i=n+1}^{l}(X^2-\epsilon_i)$ for some collection of real numbers $0<\epsilon_1 \ll \epsilon_2 \ll \ldots \ll \epsilon_l \ll 1$. The iterated vanishing cycle construction (cf. the relative vanishing cycle construction, Lemma 32

of [1]) is well defined using honest (neither deformed, nor rescaled) symplectic parallel transport along this composite of linear paths when each $\epsilon_i$ is chosen sufficiently small in terms of $\epsilon_{i-1}$ and the Kähler metric defined near $X^l \in S_l$.

To continue the construction along the rest of $\gamma_{A_0}$ and $\gamma_{B_0}$ it is in general necessary to deform symplectic parallel transport (see Remark 2.21). We now deform the pullback $\gamma_{A_0}^*\Theta_l$ on the fibration $\gamma_{A_0}^* S_l$. This deformation, performed on fibres away from a small neighbourhood of $0 \in [0,1]$, suffices to give a well-defined iterated vanishing cycle in $\mathcal{Y}_{l,P_0}$. The same works for $\gamma_{B_0}$.

We can view $K_{A_1}, K_{B_1}$ as an intermediate step in the iterated vanishing cycle construction for $\gamma_A, \gamma_B$ respectively, since these paths split with first sections $\gamma_{A_1}, \gamma_{B_1}$. One can continue the construction with deformed one forms $\Theta_{A_0}, \Theta_{B_0}$ to control symplectic parallel transport over the remainder of the paths (i.e. over $P_1\gamma_{A_0}, P_1\gamma_{B_0}$). We denote the resulting Lagrangians in the fibre $\mathcal{Y}_{k+l,P}$ by

$$L_A(\Theta_{k+l}, \Theta_{A_0}), L_B(\Theta_{k+l}, \Theta_{B_0}).$$

In general, for any choice of $\Theta$ and necessary deformations of it, the Lagrangians $L_A(\Theta, \Theta_{A_0})$, $L_B(\Theta, \Theta_{B_0})$ depend only on a neighbourhood containing the Lagrangians at each stage of the construction. Suppose for some reason that all stages of the construction, starting with $K_A$ or $K_B$, are guaranteed to remain entirely within the neighbourhood of $X^l\mathcal{Y}_{k,P_1}$ on which $\Psi$ is defined, then $\Theta$ and the deformations need only be defined in that neighbourhood of $X^l\mathcal{Y}_{k,P_1}$. Of particular interest are those $\Theta$ which respect the splitting $\mathcal{Y}_{k,P_1} \times S_l$ together with deformations which change only the $S_l$ part of $\Theta$.

To prove Theorem 5.12 one adjusts the choices of Kähler forms (this gives canonical isomorphisms on cohomology and corresponding homotopy equivalences on the cochain complexes). Then the proof proceeds in two stages. Firstly one deforms $\Theta_{k+l}$ and $\Theta_{A_0}, \Theta_{B_0}$ until the Lagrangian iterated vanishing cycles are contained inside a holomorphically convex neighbourhood on which $\Psi$ is defined. This induces a 'continuation map' on the Floer cochain complex. Then one performs a deformation of exact symplectic forms (and the various 1–forms), defined only in the convex region, ending with the pushforward under $\Psi$ of $\Theta_k + \Theta_l$ (with deformations only of the $\Theta_l$ part). This gives a second continuation map. The composite with the first continuation map is the cochain homotopy equivalence given in the Theorem. Some care is needed to show the manner in which it is 'canonical'.

*Proof of Theorem 5.12.* The slice $S_{k+l}$ can be identified with $\mathbb{C}^{4(k+l)-1}$ (see either definition) with the origin corresponding to $X^{k+l}$ and such that $X^l S_k$ is a linear subspace. We decompose $S_{k+l} \cong X^l S_k \oplus V$, where $V$ is a linear complement to $X^l S_k$. Now we choose the exhausting plurisubharmonic function $\rho_k$ on $S_k$ and any exhausting plurisubharmonic function $\rho_V$ on $V$. The only condition we impose is that each should be $||z||^2$ under some linear isomorphisms $X^l S_k \cong \mathbb{C}^{4k-1}$, $V \cong \mathbb{C}^{4l}$ respectively. This is useful to have later for calculations of deformations.

**Remark 5.14.** The condition $\rho = ||z||^2$ in some global complex coordinates is useful for calculating certain deformations. In particular, we can perform the following construction with the exact Kähler form $\Omega = -d(d\rho \circ i)$. By calculation in the given complex coordinates, we have $-d\rho \wedge (d\rho \circ i) = \rho\Omega$, and hence for any smooth function $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ it follows that $-d(d(h(\rho)) \circ i) = (h'(\rho) + \rho h''(\rho))\Omega$. Now by choosing any smooth $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, supported away from 0, and setting

$$h(t) = \int_0^t f(s)(\log t - \log s)ds$$

we find that $-d(d(h(\rho))\circ i) = f(\rho)\Omega$. Suppose we choose $f \geq 0$ everywhere, then $-d(d(h(\rho))\circ i)$ is exhausting and subharmonic.

Consider the family of plurisubharmonic functions $\rho_k + s\rho_V$ for $s > 0$. This restricts as $\rho_k$ to $X^l S_k$ so, taking $\rho_{k+l} = \rho_k + s\rho_V$ for some such $s$, the Lagrangian submanifolds $K_{A_1}, K_{B_1} \subset \mathcal{Y}_{k,P_1}$ are well defined independently of $s$. Furthermore, we can choose some $R$ (independent of $s$) such that $X^l K_{A_1}, X^l K_{B_1}$ lie strictly inside the $\rho_{k+l} = R$ level-set.

Choosing $s$ large enough, we can ensure that $\rho_{k+l}^{-1}[0, 4R]$ is confined to any particular open neighbourhood of $S_k$ in $S_{k+l}$. Now consider the local submanifolds $\Psi(\{y\} \times S_l)$ for $y$ in $\mathcal{Y}_{k,P_1}$. These cannot be tangent at $X^l y$ to $X^l S^k$ (they are not tangent to $X^l \mathcal{Y}_{k,P_1}$ and other tangencies to $X^l S_k$ would imply that their projection (Proposition 5.6) to $\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}) \times \xi$ varies in the $\xi$ direction). Hence, by compactness of $\rho_{k+l}^{-1}[0, 4R]$, we can also ensure that $\rho_{k+l}^{-1}[0, 4R] \cap \chi^{-1}(P_1 \overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}))$ is contained within an arbitrarily small neighbourhood of $X^l \mathcal{Y}_{k,P_1}$. Let $s$ be any number large enough such that this small neighbourhood lies within the range of $\Psi$.

We now choose a candidate $\tilde{\rho}_l$ for the function $\rho_l$ on $S_l$ which is $||\mathbf{z}||^2$ w.r.t. the natural linear coordinates one has from entries of the matrices $A_1 \in \mathfrak{sl}_n(\mathbb{C})$ and $A_2, \ldots, A_n \in \mathfrak{gl}_n(\mathbb{C})$ defining $S_l$. This gives an exhausting plurisubharmonic function $\tilde{\rho}_{split} := \rho_k + \tilde{\rho}_l$ on $\mathcal{Y}_{k,P_1} \times S_l$. Scaling up $\tilde{\rho}_l$ by a large enough constant factor, we can ensure that the $R$ level-set of $\tilde{\rho}_{split}$ in $\mathcal{Y}_{k,P_1} \times S_l$ maps by $\Psi$ to within the $2R$ level-set of $\rho_{k+l}$. This level-set necessarily contains $K_{A_1} \times \{X^l\}$ and $K_{B_1}) \times \{X^l\}$.

The enclosures used in the localisation argument will be $(\rho_{k+l}, 4R)$ and $(\tilde{\rho}_{split}, R)$. In the appropriate setting they will be shown to be equivalent.

Now we consider in detail the process by which the Lagrangian iterated vanishing cycles are defined, starting from $K_{A_1}, K_{B_1}$. By comparison to a region of $\mathcal{Y}_{k,P_1} \times S_l$ and choice of $\rho_{split} = \rho_l + \rho_k$, we show how to deform the exact symplectic structure on the range of $\Psi$ and how to choose the deformations controlling symplectic parallel transport, such that the iterated vanishing cycles are contained in the image under $\Psi$ of the $R$ level-set of $\tilde{\rho}_{split}$.

We begin by considering the paths $\gamma_{A_0}, \gamma_{B_0}$ in $\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C})$ and the iterated vanishing cycle construction in $S_l$ over them. We start with the exact Kähler form

$$\tilde{\Omega}_l := d\tilde{\Theta}_l := d(-d\tilde{\rho}_l \circ i)$$

on $S_l$, and with deformations $\tilde{\Theta}_{l,A_0}, \tilde{\Theta}_{l,B_0}$ of 1–forms over the vanishing paths necessary for the iterated vanishing cycle construction. In general, the Lagrangian iterated vanishing cycles in $S_l$ one defines in this way are not confined to the region where $\Psi$ is defined. We *fix* this by rescaling, using a holomorphic $\mathbb{C}^*$–action compatible with the fibration $S_l$. For $\lambda \in \mathbb{C}^*$, this acts on $S^l$ as:

$$\lambda\colon\ X^n I - \sum_{i=1}^{n} X^{n-i} A_i \longrightarrow X^n I - \sum_{i=1}^{n} X^{n-i} \lambda^i A_i$$

and on $\overline{\mathrm{Conf}}^0_{2l}(\mathbb{C})$ by multiplying all roots by $\lambda$. We will only be interested in sufficiently small positive real values of $\lambda$.

We define a new exhausting plurisubharmonic function $\tilde{\rho}_l^{\lambda}$ as the pushforward by the action of $\lambda$. Similarly, we get $\tilde{\Theta}_l^{\lambda}$ and deformations $\tilde{\Theta}_{l,\lambda(A_0)}, \tilde{\Theta}_{l,\lambda(B_0)}$. These deformations are defined over new iterated vanishing paths

$$\gamma_{\lambda(A_0)} := \lambda(\gamma_{A_0}) \qquad\qquad \gamma_{\lambda(B_0)} := \lambda(\gamma_{B_0})$$

representing the scaled crossingless matchings $\lambda(A_0), \lambda(B_0)$ respectively. The Lagrangian iterated vanishing cycles also respect this pullback, namely:

$$L_{\lambda(A_0)}(\tilde{\Theta}_l^\lambda, \tilde{\Theta}_{l,\lambda(A_0)}) = \lambda(L_{A_0}(\tilde{\Theta}_l, \tilde{\Theta}_{l,A_0}))$$
$$L_{\lambda(B_0)}(\tilde{\Theta}_l^\lambda, \tilde{\Theta}_{l,\lambda(B_0)}) = \lambda(L_{B_0}(\tilde{\Theta}_l, \tilde{\Theta}_{l,B_0}))$$

Now choose $\lambda$ small enough that the new iterated vanishing cycles are contained within so small a level set that the products $K_{A_1} \times L_{\lambda(A_0)}(\tilde{\Theta}_l^\lambda, \tilde{\Theta}_{l,\lambda(A_0)})$ and $K_{B_1} \times L_{\lambda(B_0)}(\tilde{\Theta}_l^\lambda, \tilde{\Theta}_{l,\lambda(B_0)})$ lie within the enclosure $(\tilde{\rho}_{split}, R)$ in $\mathcal{Y}_{k,P_1} \times S_l$. Now we define $\rho_l := \rho_l^\lambda$ and $\rho_{split} := \rho_l + \rho_k$. Importantly, these are plurisubharmonic for the same complex structure as $\tilde{\rho}_l, \tilde{\rho}_{split}$ respectively. This also specfies $\Theta_l$ and $\Theta_{split} = \Theta_l + \Theta_k$. Similarly, $\Theta_{split,A_0}, \Theta_{split,B_0}$ are given by adding $\Theta_k$ to the deformations $\Theta_{l,A_0}, \Theta_{l,B_0}$. These define relative vanishing cycles to $K_{A_1}, K_{B_1}$ respectively, which split the products written above. By careful isotopy of $\Theta_{k+l}$ to make it equal to $\Psi_*\Theta_{split}$ within the image under $\Psi$ of the enclosure $(\tilde{\rho}_{split}, R)$, we shall control the position of the iterated Lagrangian vanishing cycles in $S_{k+l}$ in the required manner.

It is helpful to define one more exact 2–form $\Omega_{flat} = d\Theta_{flat}$ before defining an isotopy of exact symplectic forms relating $\Omega_{k+l}$ and $\Psi_*\Omega_{split}$. For a suitable choice of smooth function $h\colon \mathbb{R} \longrightarrow \mathbb{R}$, the function $\rho_{flat} := h(\rho_{k+l})$ vanishes where $\rho_{k+l} \leq 2R$, and is plurisubharmonic elsewhere (cf. Remark 5.14). We define $\Theta_{flat} = -d\rho_{flat} \circ i$.

Let $g\colon \mathbb{R} \longrightarrow [0,1]$ be any smooth function such that $g(t) = 1$ for $t \leq 2R$ and $g(t) = 0$ for $t \geq 3R$. Then $g\Psi_*\Theta_{split}$ extends to a 1–form on all of $\chi^{-1}(P_1\overline{\mathrm{Conf}}_{2l}^0(\mathbb{C}))$. Consider the compact family
$$d(r\Theta_{k+l} + s\Theta_{flat} + t\Psi_*\Theta_{split} + u\epsilon g\Psi_*\Theta_{split})$$
of 2–forms where $r, s, t, u \in [0,1]$ and $\min\{r, s, t\} \geq \frac{1}{2}$. The space of all Kähler forms is convex, so these are Kähler for $\rho_{k+l} \leq 2R$ (and for trivial reasons for $\rho_{k+l} \geq 3R$). Non-degeneracy of 2–forms is an open condition, so there exists some $\epsilon > 0$ small enough such that the 2–forms are all symplectic.

We are now ready to define the isotopy of exact symplectic forms from $\Omega_{k+l}$ to $\Psi_*\Omega_{split}$. Namely, we take the linear isotopies between the following 1–forms (in the given order):

(1) $\Theta_{k+l}$

(2) $\Theta_{k+l} + \epsilon g\Psi_*\Theta_{split}$

(3) $\Theta_{flat} + \epsilon g\Psi_*\Theta_{split}$

(4) $\Psi_*\Theta_{split} + \epsilon g\Psi_*\Theta_{split}$

(5) $\Psi_*\Theta_{split}$

From stage 1 to stage 3, this is an isotopy of exact symplectic forms defined on all of $\chi^{-1}(P_1\overline{\mathrm{Conf}}_{2l}^0(\mathbb{C}))$ which are equal to $\Omega_{k+l}$ where $\rho_{k+l} \geq 3R$. Then it continues to stage 5 as an isotopy of forms defined only where $\rho_{k+l} < 4R$.

We choose, along with the 1-forms from stages 1 to 3, smoothly varying families of deformations over the paths $P_1\lambda(\gamma_{A_0}), P_1\lambda(\gamma_{B_0})$ to control symplectic parallel transport. These deformations are chosen to be supported where $\rho_{k+l} > 4R$. From stage 3 onwards we can use the deformations $\Theta_{split,A_0}, \Theta_{split,B_0}$ of whichever constant multiple of $\epsilon g\Psi_*\Theta_{split}$ we have where $\rho_{k+l} < 2R$.

The transition to these deformations at stage 3 is no problem. Namely, the deformations given by $\Theta_{split,A_0}, \Theta_{split,B_0}$ force all stages of the construction of the relative vanishing cycles to $K_{A_1}, K_{B_1}$ to be contained in $\rho_{k+l}^{-1}[0, 2R]$. Hence, we can remove the previous deformations

(by isotopy) without changing the Lagrangians. In fact, the relative vanishing cycles are now the images under $\Psi$ of the product Lagrangians $K_{A_1} \times L_{\lambda(A_0)}(\tilde{\Theta}_l^{\lambda}, \tilde{\Theta}_{l,\lambda(A_0)})$ and $K_{B_1} \times L_{\lambda(B_0)}(\tilde{\Theta}_l^{\lambda}, \tilde{\Theta}_{l,\lambda(B_0)})$ described earlier.

$\Theta_{flat} + \epsilon g \Psi_* \Theta_{split}$ is Kähler for the complex structure we have on $\chi^{-1}(P_1 \overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}))$ outside of the $\rho_{k+l} = 3R$ level-set, so none of the holomorphic strips defining the differential in the Floer cochain complex leaves the $\rho_{k+l} \leq 4R$ locus. For the remaining stages, the forms remain Kähler where $3R \leq \rho k + l \leq 4R$, so we can get away with calculating Floer cohomology only within this neighbourhood. Also from stage 3 onwards we always have a constant multiple of $\Psi_* \Theta_{split}$ where $\rho_{k+l} < 2R$, so we may continue to use the same deformations to control symplectic parallel transport. This means that the Lagrangian relative vanishing cycles do not change.

Finally, once we reach the end of this isotopy of exact symplectic structures, we can compare the Floer cochain complex to

$$CF^*(\Psi^{-1}(K_{A_1}) \times L_{\lambda(A_0)}(\tilde{\Theta}_l^{\lambda}, \tilde{\Theta}_{l,\lambda(A_0)}), \Psi^{-1}(K_{B_1}) \times L_{\lambda(B_0)}(\tilde{\Theta}_l^{\lambda}, \tilde{\Theta}_{l,\lambda(B_0)}))$$

within $\mathcal{Y}_{k,P_1} \times S_l$. We find that nothing has changed, since all holomorphic strips used to define the differential here lie within the $R$ level-set of $\rho_{split}$ and this is contained in the preimage under $\Psi$ of the $\rho_{k+l} \leq 4R$ region of $\chi^{-1}(P_1 \overline{\mathrm{Conf}}^0_{2l}(\mathbb{C}))$. All the data defining this splits, so the cochain complex splits as a tensor product:

$$CF^*(\Psi^{-1}(K_{A_1}), \Psi^{-1}(K_{B_1})) \otimes CF^*(L_{\lambda(A_0)}(\tilde{\Theta}_l^{\lambda}, \tilde{\Theta}_{l,\lambda(A_0)}), L_{\lambda(B_0)}(\tilde{\Theta}_l^{\lambda}, \tilde{\Theta}_{l,\lambda(B_0)}))$$

Using the continuation maps associated to the isotopies of exact symplectic structures and simultaneous isotopy of the Lagrangians, we find this is isomorphic to the following:

$$\mathrm{CKH}_{\mathrm{symp}}(P_1, A_1, B_1) \otimes \mathrm{CKH}_{\mathrm{symp}}(\lambda(P_0), \lambda(A_0), \lambda(B_0))$$

which, by the continuation map induced by varying the rescaling parameter from 1 to $\lambda$ as in Lemma 5.11, is isomorphic to

$$\mathrm{CKH}_{\mathrm{symp}}(P_1, A_1, B_1) \otimes \mathrm{CKH}_{\mathrm{symp}}(P_0, A_0, B_0)$$

It remains only to describe how these isomorphisms relate to the canonical isomorphisms on symplectic Khovanov homology (and to give the cochain level version of this). The cochain complex $CF^*(\Psi^{-1}(K_{A_1}), \Psi^{-1}(K_{B_1}))$ is canonically isomorphic to $\mathrm{CKH}_{\mathrm{symp}}(P_1, A_1, B_1)$. However, for $(P_0, A_0, B_0)$ the corresponding statement is not quite true. The $S_l$ factor in $\mathcal{Y}_{k,P_1} \times S_l$ is only identified up to some automorphism of $S_l$. Hence, $CF^*(L_{\lambda(A_0)}(\tilde{\Theta}_l^{\lambda}, \tilde{\Theta}_{l,\lambda(A_0)}), L_{\lambda(B_0)}(\tilde{\Theta}_l^{\lambda}, \tilde{\Theta}_{l,\lambda(B_0)}))$ is identified with $\mathrm{CKH}_{\mathrm{symp}}(P_0, A_0, B_0)$ up to an automorphism of cochain complexes.

At the other end of the construction we have some choice of $\mathrm{CKH}_{\mathrm{symp}}(P, A, B)$. Any two choices are related by a chain homotopy equivalence inducing the canonical isomorphisms on cohomology. The isotopy of exact symplectic structures given above induces a well defined continuation map

$$\mathrm{CKH}_{\mathrm{symp}}(P, A, B) \longrightarrow \mathrm{CKH}_{\mathrm{symp}}(P_1, A_1, B_1) \otimes \mathrm{CKH}_{\mathrm{symp}}(P_0, A_0, B_0)$$

since the space of choices involved was connected.

$\square$

**Lemma 5.15.** *Let $D$ be a disc centred at $0 \in \mathbb{C}$. Suppose we have a bridge diagram $(P, A, B)$ which splits as in Theorem 5.12 as the union of bridge diagrams $(P_1, A_1, B_1)$ supported away from 0 and $(P_0, A_0, B_0)$ supported inside $D$.*

*Let $u$ be an admissible map from a disc $B$ into $\overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$ with image contained in*

$$P_1 \,\mathrm{Conf}^0_{2l}(D) \subset \overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$$

*Suppose also that $u$ has a single input marked point mapping to $P_1P_0$ and output marked point mapping to some $P_1P_0'$. We write $(P', A', B')$ for the bridge diagram given by the output end of the saddle cobordisms induced by $u$. It is the union of $(P_1, A_1, B_1)$ with some bridge diagram $(P_0', A_0{\prime}, B_0{\prime})$ supported inside $D$.*

*Then the relative invariant factors up to homotopy through the splittings (as Künneth products) of $\mathrm{KH}_{\mathrm{symp}}$ at each end given by Theorem 5.12*

$$\begin{tikzcd}
\mathrm{KH}_{\mathrm{symp}}(P, A, B) \arrow[r, "f_u"] \arrow[d] & \mathrm{KH}_{\mathrm{symp}}(P', A', B') \arrow[d] \\
\begin{array}{c} \mathrm{KH}_{\mathrm{symp}}(P_1, A_1, B_1) \\ \otimes \\ \mathrm{KH}_{\mathrm{symp}}(P_0, A_0, B_0) \end{array} \arrow[r, "f_1 \otimes f_0"] & \begin{array}{c} \mathrm{KH}_{\mathrm{symp}}(P_1, A_1, B_1) \\ \otimes \\ \mathrm{KH}_{\mathrm{symp}}(P_0', A_0', B_0') \end{array}
\end{tikzcd}$$

*Furthermore, the bottom map splits w.r.t. this product as $f_1 \otimes f_0$ where $f_1$ is the identity on $\mathrm{KH}_{\mathrm{symp}}(P_1, A_1, B_1)$ and $f_0$ the map on $\mathrm{KH}_{\mathrm{symp}}(P_0, A_0, B_0)$ induced by considering $u$ as a map to $\mathrm{Conf}^0_{2l}(D)$.*

**Remark 5.16.** I use the same splitting to define, up to an overall sign ambiguity, chain maps induced by creation and annihilation cobordisms (see Section 5.3). Therefore, up to this sign ambiguity, the splitting of maps described above in fact holds for general cobordisms supported near $0 \in \mathbb{C}$.

*Proof.* This is an adaptation of the proof of Theorem 5.12. One begins with a tree construction (as in Section 2.2) representing $u$.

Throughout the argument one must then consider the deformations of one forms over iterated vanishing paths leading to $P = P_1P_0$ and also all discs in the tree construction. The immediate problem is that the pullback of $d(\Theta_{flat} + \epsilon g \Psi_* \Theta_{split})$ is not an allowed choice of 2–form over any of these discs containing a singular value, since it is not in general Kähler near the singular locus. The same problem occurs for other of the forms used in the argument.

Singular values are isolated in the interior of the discs, and we only need control of symplectic parallel transport around the boundaries, so we can perform the following correction.

First, choose the deformation controlling symplectic parallel transport to be supported within distance $\epsilon/2$ of the boundary of the disc. Then between distance $\epsilon/2$ and $\epsilon$ of the boundary one linearly interpolates (w.r.t. the radial coordinate on the disc) between the offending 1–form $\Theta_{flat} + \epsilon g \Psi_* \Theta_{split}$ and the 1–form $\Theta_{k+l}$.

The same trick can be performed with $\Theta_{k+l}$ replaced by $\Theta_{split}$ or any convex combination of the two. Hence the entire localisation argument of Theorem 5.12 works for tree constructions as well. □

Suppose $P = P(X) \in \overline{\mathrm{Conf}}^0_{2n}(\mathbb{C})$. We can translate all the roots of $P$ by $\mu \in \mathbb{C}$ by a change of variables replacing $P(X)$ by $P(X-\mu)$. Using this trick, we can map $\mathrm{Conf}^0_{2k}(\mathbb{C} \setminus \{(1+\frac{k}{l})\mu\})$ into $\overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$ by

$$P(X) \mapsto (X-\mu)^{2l} P(X + \frac{l}{k}\mu)$$

such that the image is precisely the subset of $\overline{\mathrm{Conf}}^0_{2(k+l)}(\mathbb{C})$ of polynomials with root $\mu$ of multiplicity $2l$.

There is a map of fibrations $S_k \longrightarrow S_{k+l}$ over this map which has a similar description. Suppose $y(X) \in S_k$, then $y(X)$ is a polynomial of degree $k$ with matrix coefficients. We define the map of fibrations as

$$y(X) \mapsto (X-\mu)^l y(X + \frac{k}{l}\mu)$$

This is an affine linear map (in terms of the entries of all the matrix coefficients of $y(X)$). Restricting to fibres of $S_k$ over $\mathrm{Conf}^0_{2k}(\mathbb{C} \setminus \{(1+\frac{k}{l})\mu\})$, this map has image in the singular locus of the fibres of $S_{k+l}$ which it hits. In fact all the calculations of this section generalise straightforwardly to this setting.

**Remark 5.17.** This has the consequence that Theorem 5.12 (and Lemma 5.15) hold when applied to unlinked components of a bridge diagram (or cobordism of bridge diagrams) supported on a small disc not necessarily centred at $0 \in \mathbb{C}$. The only extra complication is that, in decomposing a bridge diagram into two, one has to translate the resulting bridge diagrams in $\mathbb{C}$ to ensure they are of the form $(P, A, B)$ with $P \in \mathrm{Conf}^0_{2\mathrm{n}}(\mathbb{C})$.

## 5.3 Creation/annihilation cobordisms

Suppose we have two bridge diagrams $D$ and $D'$ which are everywhere identical except that $D'$ contains, in some small neighbourhood, an extra unlinked component formed from a single alpha and beta curve. By Theorem 5.12 there is an isomorphism

$$\mathrm{KH}_{\mathrm{symp}}(D') \longrightarrow \mathrm{KH}_{\mathrm{symp}}(D) \otimes H^*(S^2)$$

which is canonical on the first factor.

We now define the creation and annihilation maps explicitly in terms of this splitting. Namely the creation map, corresponding to the elementary cobordism $D$ to $D'$, shall be:

$$\begin{array}{ccc} \mathrm{KH}_{\mathrm{symp}}(D) & \longrightarrow & \mathrm{KH}_{\mathrm{symp}}(D) \otimes H^*(S^2) \\ r & \mapsto & r \otimes 1 \end{array}$$

Here we view $H^*(S^2)$ as the ring $\mathbb{Z}[X]/(X^2)$. The annihilation map corresponding to the elementary cobordism $D'$ to $D$, shall be:

$$\begin{array}{ccc} \mathrm{KH}_{\mathrm{symp}}(D) \otimes H^*(S^2) & \longrightarrow & \mathrm{KH}_{\mathrm{symp}}(D) \\ r \otimes 1 & \mapsto & 0 \\ r \otimes X & \mapsto & r \end{array}$$

In both cases, composing with a (grading preserving) automorphism of the $H^*(S^2)$ factor can only change the map by a sign. Hence they are well defined up to sign.

The motivation behind these two definitions is twofold. Firstly, they are simply defined to copy the corresponding maps between Khovanov homology groups. Secondly it is necessary in order to make the stabilisation and destabilisation maps of Section 5.4 independent of the vertex of a bridge diagram at which one performs stabilisation.

## 5.4 Stabilisation and destabilisation maps

As shown earlier, stabilisation of a bridge diagram yields a new bridge diagram and an isomorphism between the old and new symplectic Khovanov homologies. In this section we shall show that this isomorphism can be realised as the composite of a creation map and a single saddle cobordism. A similar construction will also be made for destabilisation.

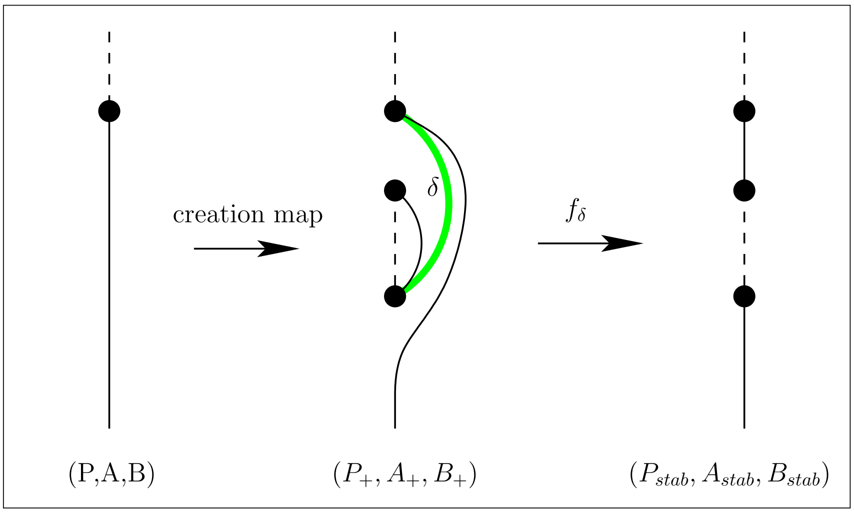


Figure 23: An illustration of the stabilisation map locally near a vertex of the bridge diagram $D$

**Definition 5.18.** Let $(P, A, B)$ be any bridge diagram. Choose a vertex $v$ and define a new bridge diagram $(P_+, A_+, B_+)$ by adding an unlinked unknot component consisting of a single $\alpha$– and $\beta$–curve both supported near $v$. Let $\delta$ be any curve supported near $v$ and joining $v$ to one of the vertices of the unknot component. Let $(P_{stab}, A_{stab}, B_{stab})$ be the bridge diagram obtained by performing the saddle cobordism which $\delta$ specifies. Figure 23 illustrates these constructions locally near $v$.

We define the *stabilisation map* to be the composite of two maps: the creation map

$$\mathrm{KH}_{\mathrm{symp}}(P, A, B) \longrightarrow \mathrm{KH}_{\mathrm{symp}}(P_+, A_+, B_+)$$

and the map induced by $\delta$

$$f_\delta \colon \mathrm{KH}_{\mathrm{symp}}(P_+, A_+, B_+) \longrightarrow \mathrm{KH}_{\mathrm{symp}}(P_{stab}, A_{stab}, B_{stab})$$

We define the *destabilisation map* as a similar composite. First one performs an elementary cobordism $\mathrm{KH}_{\mathrm{symp}}(P_{stab}, A_{stab}, B_{stab}) \longrightarrow \mathrm{KH}_{\mathrm{symp}}(P_+, A_+, B_+)$, then the annihilation map.

To study these maps I localise everything to the model neighbourhood given by Lemma 4.6. Analogously to Section 5.2, I begin by studying stabilisation supported on a neighbourhood of $0 \in \mathbb{C}$. Generalising to the case of stabilisation anywhere in the plane is a simple, but notationally intensive, application of the same method, translating configurations.

In order to phrase the effect of localisation to this model neighbourhood, we first need to consider the iterated vanishing cycle construction in compatible terms. Let $(XP(X), A, B)$ be a bridge diagram with $2m-2$ vertices, one of which lies at 0. Let $\gamma_A, \gamma_B$ be iterated vanishing paths in $\overline{\mathrm{Conf}}^0_{2m-2}(\mathbb{C})$ corresponding to the crossingless matchings $A, B$ respectively. With the appropriate choices made, these give iterated vanishing cycles $K_A, K_B \subset \mathfrak{Y}_{2m-2,XP(X)}$. These data are the startpoint of the constructions of this section.

The base $\mathbb{C}^2$ of the model neighbourhood corresponds locally near $(0,0)$ to a subset of $\overline{\mathrm{Conf}}^0_{2m}(\mathbb{C})$ by the map

$$(d, z) \mapsto (X^3 - Xd + z)P(X)$$

In particular the critical values are the points of the form $(3x^2, 2x^3)$ which correspond to configurations $(X-x)^2(X+2x)P(X)$ with a root of multiplicity 2 at $x$.

Pick some small $x$ and let $d = 3x^2$. I will discuss how small this $x$ has to be later. We now extend $\gamma_A, \gamma_B$ by composing with the path $t \mapsto (X + 2tx - \frac{tx}{m-1})P(X - \frac{tx}{m-1})$ for $t$ ranging from 0 to 1. This is the same as composing $X^2\gamma_A, X^2\gamma_B$ with the path $(X-tx)^2(X+2tx)P(X)$. Denote the composite paths by $\gamma_{A,x}, \gamma_{B,x}$. To finish the iterated vanishing cycle construction we choose vanishing paths $\psi_A, \psi_B$ in $\mathbb{C}$ for the Lefschetz fibration

$$\begin{tikzcd} \mathbb{C}^3 \arrow[d, "\pi"] & (a,b,c) \arrow[d, mapsto] \\ \mathbb{C} & a^3 - ad + bc \end{tikzcd}$$

We then compose the paths $(X^3 - Xd + \psi_A)P(X), (X^3 - Xd + \psi_B)P(X)$ with $\gamma_{A,x}, \gamma_{B,x}$ (considered as paths in $\overline{\mathrm{Conf}}^0_{2m}(\mathbb{C})$) respectively to give iterated vanishing paths $\gamma'_A, \gamma'_B$ in $\overline{\mathrm{Conf}}^0_{2m}(\mathbb{C})$ ending at some regular value $P' := (X^3 - Xd + z)P(X)$ of $\chi$.

Suppose we conclude that $x$ needs to be smaller, then this whole construction can be scaled down continuously by $(d, z) \mapsto (\lambda^2 d, -\lambda^3 z)$ for small real $\lambda$.

Let $D$ be a disc centred at $0 \in \mathbb{C}$ which contains only the root 0 of $XP$ and only one section of a curve in each of $A, B$. The iterated vanishing paths $\gamma'_A, \gamma'_B$ come from a bridge diagram $(P', A', B')$ with the following properties:

- $P' = (X^3 - Xd + z)P(X)$ has three roots in $D$.
- $(P', A', B')$ is identical to $(XP, A, B)$ outside of $D$.
- $A', B'$ are each disjoint from $D$, except in one whole curve and one segment of another.

In fact, by considering the effect of wrapping $\psi_A, \psi_B$ around the two singularities $\pm 2x^3$ of the Lefschetz fibration, one sees that all $(P', A', B')$ with the above properties (for a fixed $(d, z)$) are realised in this manner. Also, any two versions of the construction for the same $(P', A', B')$ (up to isotopy supported on $D$) are essentially the same. i.e. the pairs $\psi_A, \psi_B$ are isotopic.

Corresponding to a construction as above, we define $L_A, L_B$ to be the vanishing cycles for $\psi_A, \psi_B$. If we use the standard exact Kähler form on $\mathbb{C}^3$, then for small enough $x$ this vanishing cycle construction is not only well defined, but is confined to an arbitrarily small

neighbourhood of $0 \in \mathbb{C}^3$. To see this, I refer the reader to the explicit study of the symplectic parallel transport vectors preserving the locus $\{|b| = |c|\}$ in Section 3.2. In particular one has

$$\left|\frac{\partial}{\partial t}a\right| \leq \frac{\dot{\psi}_A(t)}{3\,|a^2 - x^2|}$$

And as one scales down the choice of $x$ the lengths of the paths $\psi_A, \psi_B$ one needs decrease linearly, so this bound is sufficient to contain the vanishing cycles near $a = x$. This in itself actually has the consequence of containing it near $a = x, b = c = 0$.

**Lemma 5.19.** *Let $D$ be a disc in $\mathbb{C}$ containing 0. Let $(XP, A, B)$ and $(P', A', B')$ be bridge diagrams with the following properties:*

- *$P$ has no roots in $D$.*
- *$P' = (X^3 - Xd + z)P(X)$ has three roots in $D$.*
- *$(P', A', B')$ is identical to $(XP, A, B)$ outside of $D$.*
- *$A', B'$ are each disjoint from $D$, except in one whole curve and one segment of another.*

*Then there is an isomorphism:*

$$\mathrm{KH}_{\mathrm{symp}}(P', A', B') \longrightarrow \mathrm{KH}_{\mathrm{symp}}(XP, A, B) \otimes HF(L'_A, L'_B)$$

*which is canonical up to automorphism of the $HF(L_A, L_B)$ factor.*

*Proof.* The proof is by essentially the same method as the proof of Theorem 5.12. However, some of the initial inputs are different and not obvious, so I shall explain in detail how to overcome these differences.

We shall be using the model neighbourhood of Lemma 4.6, so here it is again as a reminder.

$$\begin{tikzcd}
\chi^{-1}(\mathbb{D}^2) \arrow[rr, "\text{local isomorphism near } X^2\mathcal{Y}_{m-1,P}\cap\chi^{-1}(0,0)"] \arrow[d, "\chi"] & & (\mathcal{F} \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4 \arrow[d, "f"] \\
\mathbb{D}^2 \arrow[rr] & & \mathbb{C}^2
\end{tikzcd}$$

We start with a study of the singular locus of the fibration $S_m$. Consider the map $\mathbb{C} \times S_{m-1} \longrightarrow S_m$ given by

$$(\mu, y(X)) \mapsto (X - \mu)y(X + \frac{\mu}{m-1})$$

This has an affine linear image which I shall denote $\mathbb{C}(S_{m-1})$. The map itself is affine linear in the $S_{m-1}$–factor, but not in the $\mathbb{C}$–factor. Furthermore, $\mathbb{C}(S_{m-1})$ contains the entire singular locus of $S_m$. To see this one uses the definition of $S_m$ as consisting of $2m \times 2m$–matrices and the fact that the singular locus is the set of $y \in S_m$ which have an eigenvalue of multiplicity 2 (cf. Lemma 24 of [1]).

We now consider $S_m$ as a vector space, with linear subspace $\mathbb{C}(S_{m-1})$. Let $\rho_m$ be a plurisubharmonic function on $S_m$ which is $||z||^2$ for under some linear isomorphism with $\mathbb{C}^{4m-1}$. The Lagrangians $K_A, K_B \subset \mathcal{Y}_{m-1,XP}$ defined by $(XP, A, B)$ are compact, so we can choose $R > 0$ such that their images $XK_A, XK_B$ in $S_m$ lie inside the $R$ level-set of $\rho_m$. By scaling up $\rho_m$

linearly in directions transverse to $\mathbb{C}(S_{m-1})$ we may also assume that $\rho^{-1}[0,4R]$ is contained in the model neighbourhood.

We can now define $\rho_{flat}$ as before to be such that $-d(d\rho_{flat}\circ i)$ is $-d(d\rho_m \circ i)$ where $\rho_m \geq 3$, is Kähler where $\rho_m > 2$ and is identically zero where $\rho_m \leq 2$. $\rho_{flat}$ is defined to be $h \circ \rho_m$ for a suitably convex $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as in Remark 5.14.

The remaining tool we need is $\Omega_{split}$, an exact Kähler form on a neighbourhood of the zero-section of $(\mathcal{F} \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4$ which splits appropriately. This is the *associated form* $\Omega$ constructed in Section 4.3 of [1].

One now proceeds with the same localisation argument as before. It is in fact slightly simpler, since, for $x$ small enough, one doesn't need deformations to control symplectic parallel transport in the $\mathbb{C}^4$–directions.

The result of this localisation argument is an isomorphism to the Floer cohomology computed in the model neighbourhood $(\mathcal{F} \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4$. The Lagrangian relative vanishing cycles obtained from $K_A, K_B$, which one uses here, split in the sense that they are given as

$$(\mathcal{F} \setminus 0)|_{K_A} \times_{\mathbb{C}^*} L'_A$$

$$(\mathcal{F} \setminus 0)|_{K_B} \times_{\mathbb{C}^*} L'_B$$

for $S^1$–equivariant Lagrangian submanifolds $L'_A, L'_B$ of the relevant fibre of $\mathbb{C}^4$. In particular this is the setting of the "non-standard splitting" dealt with in Section 3. By Remark 2.30, we can phrase the differential of Floer cohomology in terms of relative invariants. Hence, Corollary 3.1 together with the regular $S^1$-equivariant almost complex structures exhibited for this particular case in Section 3.1, gives an isomorphism from this local Floer cohomology to $\mathrm{KH}_{\mathrm{symp}}(XP, A, B) \otimes HF(L'_A, L'_B)$. □

For the stabilisation map, we are particularly interested in the case where $(P', A', B')$ is just $(XP, A, B)$ with a single unlinked unknot component added. i.e. where it is just $(P_+, A_+, B_+)$ as described in Figure 23. In this case, Theorem 5.12 already gives a splitting of the symplectic Khovanov homology. Actually, this is the same splitting as that given above in Lemma 5.19.

**Lemma 5.20.** *Let $D$ be a disc in $\mathbb{C}$ containing 0. Let $(XP, A, B)$ and $(P', A', B')$ be bridge diagrams with the following properties:*

- *$P$ has no roots in $D$.*
- *$P' = (X^3 - Xd + z)P(X)$ has three roots in $D$.*
- *$(P', A', B')$ is identical to $(XP, A, B)$ outside of $D$.*
- *$A', B'$ both have the same curve (and part of another) supported in $D$.*

*This means that $(P', A', B')$ contains a single extra unlinked unknot supported in $D$.*

*In this case $HF(L_A, L_B) \cong H^*(S^2)$. Furthermore, the splitting of $\mathrm{KH}_{\mathrm{symp}}(P', A', B')$ as a tensor product $\mathrm{KH}_{\mathrm{symp}}(XP, A, B) \otimes H^*(S^2)$ given by Lemma 5.19 is identical to that given by Theorem 5.12.*

*Proof.* One constructs $\psi_A = \psi_B$ to be short enough that the Floer cohomology calculation can be simultaneously localised in both ways with the same $\mathcal{Y}_{m-1,XP}$ factor. Then a deformation between the two models in a small enough neighbourhood gives an automorphism of $\mathrm{KH}_{\mathrm{symp}}(XP, A, B) \otimes H^*(S^2)$ which is necessarily the identity map on the first factor. □

As in Section 5.2, the splitting of symplectic Khovanov homology extends to the relative invariants from admissible maps with image in the model neighbourhood in $\overline{\mathrm{Conf}}^0_{2m}(\mathbb{C})$.

**Lemma 5.21.** *Let $u : \overline{\mathbb{D}} \longrightarrow \{(X^3 - Xd + z)P(X) : z \in \mathbb{C}\} \subset \overline{\mathrm{Conf}}^0_{2m}(\mathbb{C})$ be an admissible map. Let $(P', A', B')$ be a bridge diagram for which the Lemma 5.19 gives a splitting:*

$$\mathrm{KH}_{\mathrm{symp}}(P', A', B') \longrightarrow \mathrm{KH}_{\mathrm{symp}}(XP, A, B) \otimes HF(L_A, L_B)$$

*Then $u$ the map $f_u$ induced by $u$ on $\mathrm{KH}_{\mathrm{symp}}(P', A', B')$ splits as:*

$$\begin{tikzcd}
\mathrm{KH}_{\mathrm{symp}}(P', A', B') \arrow[r, "f_u"] \arrow[d] & \mathrm{KH}_{\mathrm{symp}}(P'', A'', B'') \arrow[d] \\
\begin{array}{c}\mathrm{KH}_{\mathrm{symp}}(P, A, B) \\ \otimes \\ HF(L'_A, L'_B)\end{array} \arrow[r, "\mathrm{id} \otimes f_{u/P}"] & \begin{array}{c}\mathrm{KH}_{\mathrm{symp}}(P'', A'', B'') \\ \otimes \\ HF(L''_A, L''_B)\end{array}
\end{tikzcd}$$

*where $f_{u/P} : HF(L'_A, L'_B) \longrightarrow HF(L''_A, L''_B)$ is the map between Floer cohomology groups induced by the admissible map $\frac{u}{P}$.*

*Proof.* This is an extension of the argument proving Lemma 5.19 in the same manner as Lemma 5.15 extends Theorem 5.12. In fact it is easier, since in the model neighbourhood one has a priori control over symplectic parallel transport. $\square$

This Lemma allows us to explicitly calculate the relative invariant used in the stabilisation map in terms of a much simpler relative invariant in the fibration $\mathbb{C}^3 \longrightarrow \mathbb{C}$. The relevant calculations are performed in Section 3.3. Lemma 3.10 describes the map

$$f_{u/P} : HF(L'_A, L'_B) \longrightarrow HF(L''_A, L''_B)$$

used in the splitting of the stabilisation. Using the definition of the creation map in terms of a splitting of symplectic Khovanov homology (together with Lemma 5.20), the stabilisation map splits as the identity map on $\mathrm{KH}_{\mathrm{symp}}(XP, A, B)$ and a composite of maps $\mathbb{Z} \longrightarrow \mathbb{Z}[X]/(X^2) \longrightarrow \mathbb{Z}$, which is the identity up to sign.

In particular, if we use the splitting just defined

$$\mathrm{KH}_{\mathrm{symp}}(XP, A, B) \cong \mathrm{KH}_{\mathrm{symp}}(XP, A, B) \otimes \mathbb{Z} \cong \mathrm{KH}_{\mathrm{symp}}(P_{stab}, A_{stab}, B_{stab})$$

the stabilisation map is the identity map on $\mathrm{KH}_{\mathrm{symp}}(XP, A, B)$. This splitting, by its construction is actually the same isomorphism originally given between the symplectic Khovanov homology of a link and of its stabilisation.

**Remark 5.22.** The same is true for the destabilisation (constructed analogously to the stabilisation map) map with respect to the same splittings. Consequently the stabilisation and destabilisation maps at a given vertex are inverses of each other.

## 5.5 The "switching move"

The two cobordisms specified locally by Figure 24 are isotopic as link cobordisms. However, the curves specifying them are not isotopic. They are related by a simple "switching move" simultaneously sliding the curve $\delta$ along the two $\alpha$–curves with which it shares vertices.

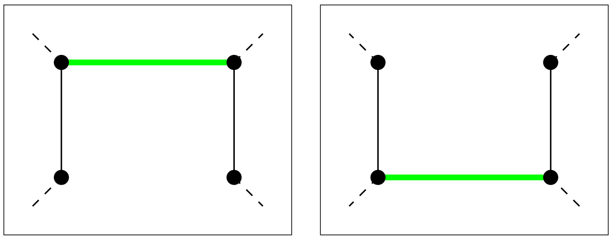

Figure 24: Two ways of performing the same cobordism of admissible links locally with non-isotopic curve $\delta$ (indicated by grey lines)

**Proposition 5.23.** *The two cobordisms specified locally by Figure 24 induce the same maps (up to sign) on* $\mathrm{KH}_{\mathrm{symp}}$.

We start with a weaker result in a very special case.

**Lemma 5.24.** *In the case specified locally by Figure 25, the two cobordisms induce the same maps (up to sign) on the* $\mathbb{Z}$ *summand in the bottom degree of cohomology.*

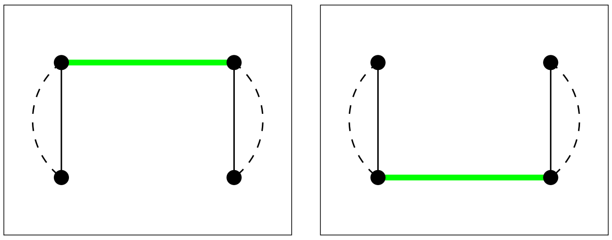

Figure 25: A trivial case of the switching move

*Proof.* Suppose first that we are dealing with the case specified globally by Figure 25.

The domain of the maps is $H^*(S^2)^{\otimes 2}$ which has a single $\mathbb{Z}$ summand in the top degree of the cohomology. Both maps are induced by the saddle cobordism part of different stabilisations, so by the previous section we know that there is, for each, a splitting in which they can be written (up to sign) as:

$$\begin{array}{rcl} H^*(S^2) \otimes \mathbb{Z}[X]/(X^2) & \longrightarrow & H^*(S^2) \otimes \mathbb{Z} \\ a \otimes 1 & \mapsto & a \otimes 1 \\ a \otimes 1 & \mapsto & 0 \end{array}$$

However, the splitting needed in either case need not necessarily be the same.

In particular, this means that they induce isomorphisms between the copies of $\mathbb{Z}$ in the bottom degree of cohomology on both sides. Up to a sign ambiguity there is only one such isomorphism, so we are done.

In the slightly less trivial case where Figure 25 is only the local model we denote by $D$, the bridge diagram for the rest of the link. The maps on symplectic Khovanov homology split (using Lemma 5.15) as the identity on $\mathrm{KH}_{\mathrm{symp}}(D)$ tensored with the maps $H^*(S^2)^{\otimes 2} \longrightarrow H^*(S^2)$ compared in the lemma. $\square$

Now, to prove Proposition 5.23, we refer to Figure 26. The figure has four numbered rows, each of which describes a cobordism composed of three elementary saddle cobordisms. The thicker grey lines indicate the cobordism to be performed as you pass to the next diagram on that row. Several of the diagrams have two cobordisms marked, but only when the lines specifying them do not intersect, so by Lemma 5.4 it does not matter in which order they are performed.

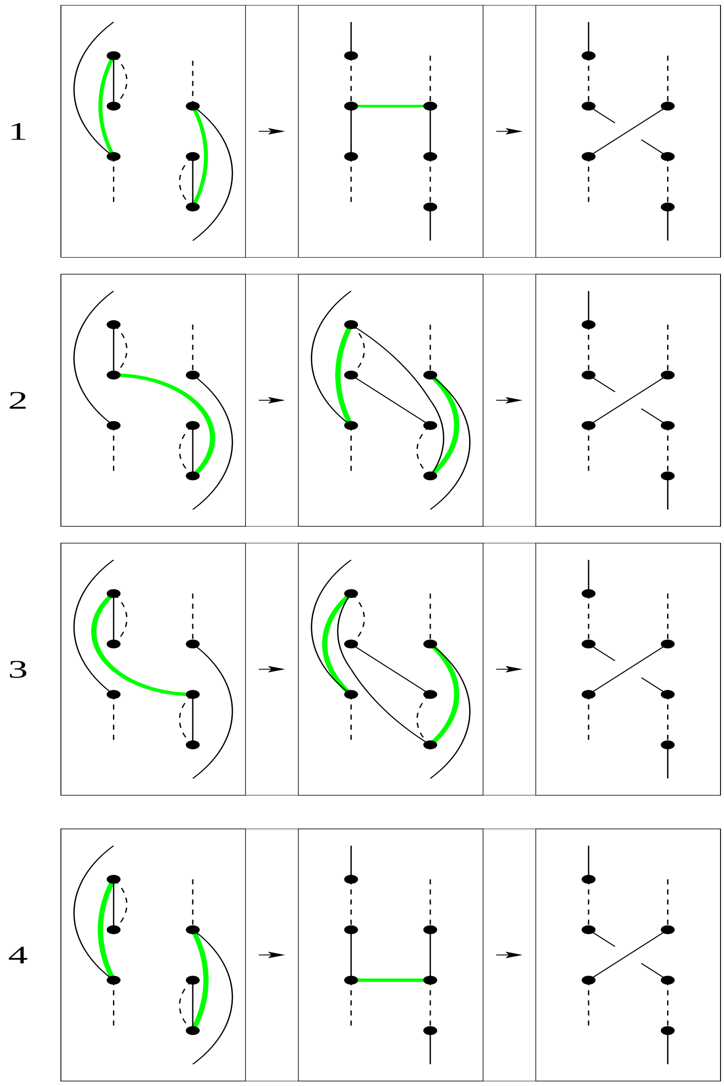


Figure 26: Four related cobordisms used to prove the invariance under the switching move.

We start with row 1. The first map is the composite of two of the saddle cobordisms used in stabilisation. The symplectic Khovanov homology splits as $\mathrm{KH}_{\mathrm{symp}}(D) \otimes H^*(S^2)^{\otimes 2}$ and all terms map to zero, with the exception of $\mathrm{KH}_{\mathrm{symp}}(D) \otimes 1 \otimes 1$ which maps isomorphically onto the symplectic Khovanov homology $\mathrm{KH}_{\mathrm{symp}}(D)$ of the next diagram. The second map in this row is the cobordism for which we would like to prove the proposition. The final diagram in row 1 represents the output end of that cobordism. I have taken the liberty of choosing an admissible position which is not quite a bridge diagram, purely because it looks simpler

(observe there is a crossing between two $\alpha$–curves).

Row 2 is an application of Lemma 5.4 to swap the order of the cobordisms in row 1. Hence, row 2 gives the same map on symplectic Khovanov homology.

Row 3 is the same as row 2 except that the switching move has been applied to the first cobordism. By Lemma 5.24 rows 2 and 3 give the same map (up to sign) on $\mathrm{KH}_{\mathrm{symp}}(D)\otimes 1\otimes 1$.

Row 3 is an application of Lemma 5.4 to row 4, just as row 2 was to row 1. Hence rows 3 and 4 give the same map.

In conclusion, rows 1 and 4 restrict to the same map (up to sign) on $\mathrm{KH}_{\mathrm{symp}}(D)\otimes 1\otimes 1$. Also the first map in both rows is identical. Since $\mathrm{KH}_{\mathrm{symp}}(D)\otimes 1\otimes 1$ maps isomorphically onto the symplectic Khovanov homology of the second diagram, it must be the case that the last maps in rows 1 and 4 are the same up to sign. This concludes the proof of Proposition 5.23.

## 5.6 Symplectic Khovanov homology of crossing diagrams

In this section I explain how $\mathrm{KH}_{\mathrm{symp}}$ of a crossing diagram is well defined, with canonical isomorphisms, up to a possible overall sign ambiguity. To do this one defines $\mathrm{KH}_{\mathrm{symp}}$ of a crossing diagram to be $\mathrm{KH}_{\mathrm{symp}}(D)$ for any bridge position which projects to that crossing diagram. The difficulty is in specifying a consistent choice of canonical isomorphism between the symplectic Khovanov homologies of any two bridge diagrams with the same projection.

Two such bridge diagrams are related by

- a sequence of stabilisations and destabilisations preserving the projection
- isotopy sliding the $\alpha$– and $\beta$–curves and vertices along the projection.

We take the stabilisation and destabilisation maps and the maps induced by these isotopies to be the canonical isomorphisms.

**Lemma 5.25.** *For a crossing diagram in which each component of the link is involved in at least one crossing the symplectic Khovanov homology is well defined (up to sign). Namely, the isomorphisms mentioned above are consistent.*

*Proof.* The condition on components being involved in crossings means that the diagram can be decomposed into edges (by cutting both strands at each crossing) without any closed loops remaining. Any bridge diagram projecting to the crossing diagram will have some number of vertices on each edge (none are possible at the crossings). Any two such bridge diagrams are isotopic through such diagrams if and only if these numbers agree on each edge. Moreover, any two such isotopies are isotopic (the crossing condition is vital here) so the maps they induce are the same.

Only the stabilisation and destabilisation maps can change the numbers of vertices on an edge. By applying the switching move (Proposition 5.23) to the saddle cobordism of a stabilisation it is immediate that it makes no difference (up to sign) at which vertex one stabilises or destabilises. It therefore suffices to show that a stabilisation followed by a destabilisation at the same vertex gives the identity map. This is covered by Remark 5.22. ☐

Careful consideration of the switching move shows also that the isotopy which moves two vertices from one edge past a crossing to another using the passing move is the same as the map that destabilises one edge and stabilises the next. Hence we can add these isotopies to the choice of canonical isomorphisms. This allows the condition that each component is involved in at least one crossing to be lifted.

A consequence is that, given a neighbourhood in which an elementary saddle cobordism is performed to a crossing diagram, it does not matter which of the curves representing a saddle cobordism one chooses on a bridge diagram to define the map on symplectic Khovanov homology. Hence the maps from creation, annihilation and saddle cobordisms (and similarly isotopy) are well defined (up to sign) on the symplectic Khovanov homology of a crossing diagram.

### 5.7 General smooth cobordisms

Given any smooth cobordism between links in $\mathbb{R}^3 \times [-1, 1]$, considerations in Morse Theory show that it can be put in a position such that the functional assigning to each point on the cobordism the value of the time ($t$–coordinate on $[-1, 1]$) has only non-degenerate critical points. Hence any cobordism is a composite of finitely many saddle cobordisms and creation/annihilation cobordisms.

In [21] cobordisms are further reduced into 'movies', i.e. finite sequences of the basic cobordisms and the Reidemeister moves on crossing diagrams. A list of 'movie moves' is given which relate isotopic cobordisms, such that any two movies of isotopic cobordisms are related by a finite sequence of these movie moves.

The movie moves (with the enumeration I shall refer to) are illustrated on pages 1221 and 1222 of Jacobsson's paper [7]. In the construction of invariants of link cobordisms for combinatorial Khovanov homology, there is a lot of work required in checking invariance under the moves. However, in the case of symplectic Khovanov homology, invariance under most of the movie moves is immediate, since they compare isotopic isotopies of links and hence induce the same maps on $\mathrm{KH}_{\mathrm{symp}}$. It is possibly that a more intrinsically geometric definition of the maps induced by cap/cup cobordisms would deal with the remaining moves. An alternative definition of the maps induced by cobordisms was recently given by Rezazadegan [9]. However, it defines the maps induced by cap/cup cobordisms in essentially the same manner.

We now prove invariance under those movie moves, for which it is not immediatly obvious by the above comments.

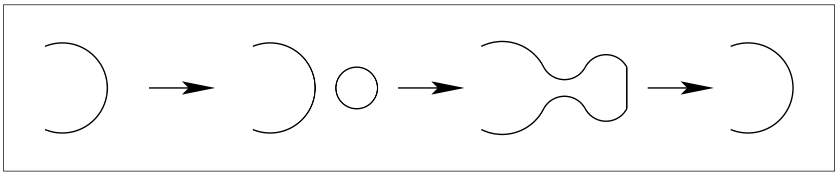

Figure 27: A non-trivial position of the trivial cobordism

As a warm-up, there is movie move 9, also known as the "castration move", which relates the cobordism in Figure 27 to the trivial one. This is realised by a stabilisation map. Hence, by definition of the canonical isomorphisms, it gives the identity map.

Moves 10 and 12 are more interesting. They are illustrated in Figures 28 as sequences of bridge diagrams with thicker lines marking the saddle cobordism still to be performed. These lines are removed once the cobordism is performed. The cobordisms should be read from top to bottom in each column.

Movie move 10 is covered by isotopies of admissible links together with simultaneous isotopy of the curve $\delta$ defining a saddle cobordism, which, by the discussion in section 5.1 proves

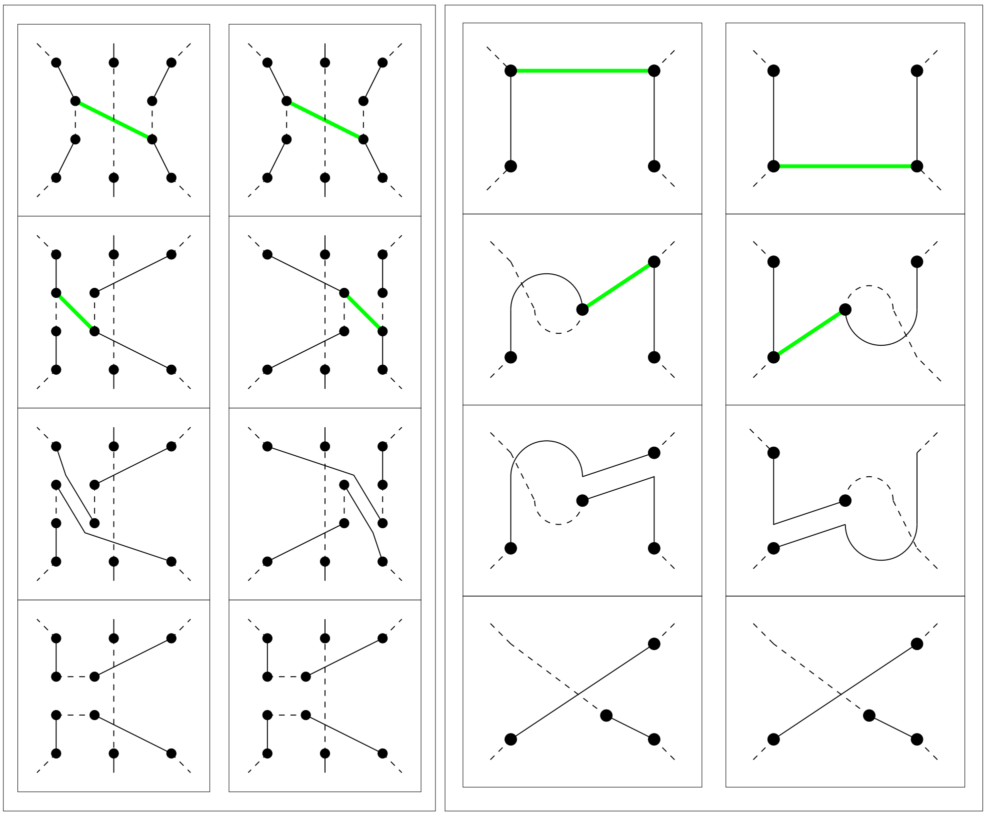

Figure 28: Movie moves 10 (between the two movies on the left) and 12 (between the two movies on the right) in terms of bridge diagrams. Each movie should be read from to to bottom. The thicker lines indicate the cobordisms still to be performed in that movie.

invariance of the induced map (see Figure 28). Movie move 12 requires, on top of that, the use of Proposition 5.23 (see the first part of the movie).

We should also prove invariance under reflected and reversed movie moves, but the proofs are identical.

As well as being invariant under the movie moves, an invariant of cobordisms up to isotopy must be invariant under the commuting of 'distant' cobordisms (i.e. supported on disjoint neighbourhoods of the diagram). This is immediate from Lemma 5.4.

In conclusion, the maps (up to sign) on the symplectic Khovanov homology of a link, induced by specific positions of link cobordisms, are actually invariants of the isotopy class of smooth cobordism. This concludes the proof of Theorem 1.1.

# A Symplectic flattening of exact Lefschetz fibrations

In this Appendix, we perform a calculation similar to Lemma 1.10 of [2]. Namely we show that, for particular exact MBL–fibrations (with isolated singular points, i.e. exact Lefschetz fibrations) we can perform a deformation in some convex region which preserves non-negativity of symplectic curvature and removes all symplectic curvature away from a certain 'vanishing locus'.

Let $(E, \pi, \Omega, \Theta, J_0, \overline{\mathbb{D}}, j)$ be an exact MBL–fibration over the unit disc (with marked points on $\partial\overline{\mathbb{D}}$) which has only one isolated singular point in the fibre over 0. Suppose furthermore, that $J_0$ can be defined everywhere, making $\rho\colon E \longrightarrow \mathbb{R}^{\geq 0}$ plurisubharmonic everywhere and exhausting in each fibre.

For large enough $R_1 \in \mathbb{R}$ we have that $\rho^{-1}[0, R_1)$ contains $\mathrm{Crit}(\pi)$ and all critical points of $\rho$. Suppose also, that we have exact Lagrangian submanifolds specified in some fibres of the boundary with the property that their images under symplectic parallel transport over paths of length at most $2\pi$ all lie in $\rho^{-1}[0, R_1)$ (and in particular are well-defined).

For $\epsilon > 0$ small enough $\rho^{-1}[0, R_1)$ also contains the set $\Sigma$ of points which are carried to the critical point by symplectic parallel transport in the radial direction from fibres over the disc $\epsilon\overline{\mathbb{D}}$ of radius $\epsilon$. We refer to $\Sigma$ as the *radial vanishing locus*. By replacing $\overline{\mathbb{D}}$ by $\epsilon\overline{\mathbb{D}}$ (and pulling back $E$ by the inclusion map) we can ensure that $\Sigma$ is well defined over the entire disc.

One now chooses $R_3 > R_2 > R_1$ and (possibly replaces $\epsilon$ with a smaller value) such that the following holds (see Figure 29):

- the image $M$ of radial symplectic parallel transport over $\epsilon\overline{\mathbb{D}}$ applied to $\pi^{-1}(0) \cap \rho^{-1}[0, R_2)$ contains $\rho^{-1}[0, R_1]$
- $\overline{M}$ is also contained in $\rho^{-1}[0, R_3)$

In fact it suffices to choose $\epsilon$ small enough given any choice of the other numbers.

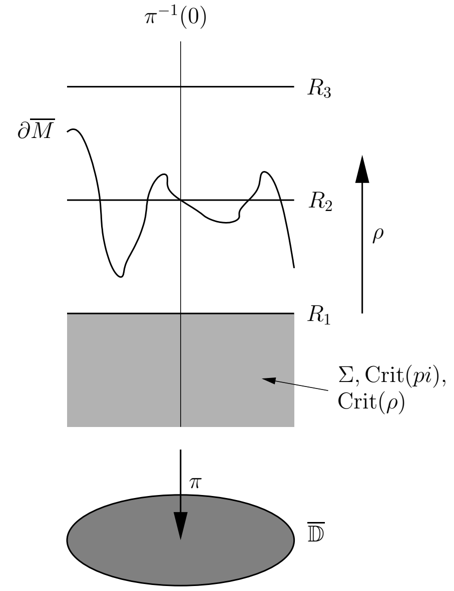


Figure 29: An illustration of the level sets of $R_1, R_2, R_3$ of $\rho$ relative to $M, \Sigma, \mathrm{Crit}(\pi), \mathrm{Crit}(\rho)$.

In replacing $\overline{\mathbb{D}}$ by $\epsilon\overline{\mathbb{D}}$, one also has to change the marked points and the exact Lagrangian boundary condition. One can think of this as replacing the fibration by a composite of fibrations of which only the pullback over $\epsilon\overline{\mathbb{D}}$ is singular. It is easily ensured that extending the Lagrangian boundary condition around the boundaries of this composite carries no Lagrangian along a path of length greater than $2\pi$. One can also view this as a smooth deformation of the fibration over $\mathbb{D}$, so the relative invariant will only be changed by composition with certain continuation maps. This is the approach we shall take, since then we do not need to explicitly study the relative invariant over the non-singular fibrations that arise in the composite.

The significance of these levels will be as follows. $\rho = R_3$ describes a convex level-set which is used as an enclosure for the Lagrangian boundary conditions. $R_2$ defines the set $M$ by radial symplectic parallel transport of $\pi^{-1}(0) \cap \rho^{-1}[0, R_2)$. This set contains the $R_1'$ level of $\rho$ and is contained within the $R_3'$ level for some $R_1', R_3'$ such that $R_3 > R_3' > R_2 > R_1' > R_1$. We perform a deformation of the exact symplectic structure supported in $M$ to make it respect a trivialisation given by radial symplectic parallel transport of a chosen subset of $\pi^{-1}(0) \cap \rho^{-1}[0, R_2)$.

**Lemma A.1.** *Suppose $\pi : E \longrightarrow \overline{\mathbb{D}}$ is an exact MBL–fibration with $\rho, R_1 < R_2 < R_3$ as constructed above. There is a deformation of $(E, \pi)$ (through exact MBL–fibrations) supported within $\rho^{-1}[0, R_3]$ preserves non-negativity of the symplectic curvature and makes the symplectic curvature inside the $R_1$ level-set vanish away from an arbitrarily small neigbourhood of the* radial vanishing locus $\Sigma$*.*

*Proof.* We take radial polar coordinates $r, \alpha$ on $\overline{\mathbb{D}}$.

The radial vanishing locus $\Sigma$ as defined earlier is smooth away from the singular point. Let $T$ be the singular fibre of $E$ minus the singular point. On $T$ we take the restrictions of the exact symplectic structure from $E$. Radial symplectic parallel transport gives a diffeomorphism

$$\Phi\colon\ M \setminus \Sigma \longrightarrow T \times \overline{\mathbb{D}}$$

Let $\Omega^T$, $\Theta^T$ be the forms on $M \setminus \Sigma$ coming from the exact symplectic structure on $T$ and the trivialisation $\Phi$. This trivialisation of the exact symplectic structure does not extend over $\Sigma$, but is useful away from $\Sigma$. We shall now use coordinates from the trivialisation.

We can split $T(M \setminus \Sigma)$ into integrable vertical and horizontal subbundles. For the purposes of this proof we shall refer to *vertical* and *horizontal* in terms of this trivialisation and not the symplectic connection. Horizontal vectors for the symplectic connection are referred to as *symplectic parallel transport vectors*. Then $d$ and $\Theta$ split as $d^V + d^H$ and $\Theta^V + \Theta^H = \Theta^T + \phi + \Theta^H$ respectively (where $\Theta^V$, $\Theta^T$ and $\phi$ vanish on horizontal tangents and $\Theta^H$ on vertical tangents w.r.t. the trivialisation $\Phi$). Since symplectic parallel transport maps fibres to fibres by symplectomorphisms we have $d^V\Theta^V = \Omega^T = d^V\Theta^T$ so $d^V\phi = 0$. Using this $\Omega$ splits as follows:

$$\Omega = d\Theta = \Omega^T + d^H\phi + d^V\Theta^H + d^H\Theta^H \tag{2}$$

The vector $\frac{\partial}{\partial r}$ lifted from $\overline{\mathbb{D}}$ is a symplectic parallel transport vector by definition of the trivialisation $\Phi$ so $\iota_{\frac{\partial}{\partial r}}\Omega$ vanishes on vertical tangents. Applying equation (2) gives $\frac{\partial \phi}{\partial r} = d^V\Theta^H(\frac{\partial}{\partial r})$ so we can evaluate $\phi$ by integrating over $r$ (for fixed $\alpha$). This gives us an expression for $\phi$ as exact on fibres:

$$\phi = \int d^V\Theta^H(\frac{\partial}{\partial r})dr = d^V\left(\int \Theta^H(\frac{\partial}{\partial r})dr\right) =: d^V h$$

So we can write the splitting of $\Theta$ as

$$\Theta = \Theta^T + \phi + \Theta^H = \Theta^T + d^V h + \Theta^H = \Theta^T + dh + (\Theta^H - d^H h)$$

where $\sigma := \Theta^H - d^H h$ vanishes on vertical tangents and also on $\frac{\partial}{\partial r}$ by our choice of definition of $h$.

Using cartesian coordinates $(x, y) = (rcos\alpha, rsin\alpha)$ on $\overline{\mathbb{D}}$ we write $\sigma = adx + bdy$. Vanishing of $\sigma$ on $\frac{\partial}{\partial r}$ gives $0 = \sigma(r\frac{\partial}{\partial r}) = adx + bdy(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) = ax + by$. Hence $a = 0$ wherever $y = 0$,

so we have smooth $\beta$ with $a = -\beta y$. We now have $\sigma = -\beta y dx + \beta x dy = \beta r^2 d\alpha$. i.e. $\Theta = \Theta^T + dh + \beta r^2 d\alpha$ and $\Omega = \Omega^T + d^V\beta \wedge r^2 d\alpha + d^H(\beta r^2 d\alpha)$. The reason we have taken so much care so far was to ensure that $\beta r^2 d\alpha$ is a smooth 1-form on $M \setminus \Sigma$.

Now it is easy to see where $\frac{\partial}{\partial\alpha}$ is a symplectic parallel transport vector. Namely, we need precisely $d^V\beta = 0$. Using the volume form $rdrd\alpha$ on $\overline{\mathbb{D}}$, a general expression for the symplectic curvature (where $U$ is vertical $\frac{\partial}{\partial r}$ and $U + \frac{\partial}{\partial\alpha}$ are symplectic parallel transport vectors) is:

$$\begin{aligned} K &:= r^{-1}\Omega(\frac{\partial}{\partial r}, U + \frac{\partial}{\partial\alpha}) \\ &= r^{-1}\left(-d^V\beta(U)r^2 d\alpha(\frac{\partial}{\partial r}) + d^H(\beta r^2 d\alpha)(\frac{\partial}{\partial r}, \frac{\partial}{\partial\alpha})\right) \\ &= r\frac{\partial}{\partial r}\beta + 2\beta \end{aligned}$$

In the singular fibre we define a smooth bump function $\psi$ such that:

- $\psi$ takes values in $[0,1]$ only
- $\psi = 0$ on a very small neighbourhood of the singular point and of $\partial\overline{M} \cap T$
- $\psi = 1$ except on some slightly larger neighbourhood of the singular point and of $\partial\overline{M} \cap T$

Then we extend $\psi$ over $M \setminus \Sigma$ by symplectic parallel transport in the $r$–direction. This gives $d^H\psi = 0$. Now we are ready to deform $\Theta$ to make $E$ trivial on $\psi^{-1}(1)$.

Let $\tilde{\Theta} := \Theta - \psi\beta r^2 d\alpha$. This extends to a smooth 1-form on all of $E$ since both summands do. It clearly satisfies the conditions just mentioned to make $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial\alpha}$ symplectic parallel transport vectors and give zero curvature on $\psi^{-1}(1)$.

Furthermore, the calculation for the symplectic curvature $\tilde{K}$ of $\tilde{\Omega} = d\tilde{\Theta}$ (everywhere, not just on $\psi^{-1}(1)$) gives:

$$\begin{aligned} \tilde{K} &:= r^{-1}d\tilde{\Theta}(\frac{\partial}{\partial r}, \tilde{U} + \frac{\partial}{\partial\alpha}) \\ &= r^{-1}\left(d^V((1-\psi)\beta r^2 d\alpha)(\frac{\partial}{\partial r}, \tilde{U}) + d^H((1-\psi)\beta r^2 d\alpha)(\frac{\partial}{\partial r}, \frac{\partial}{\partial\alpha})\right) \\ &= (1-\psi)(r\frac{\partial}{\partial r}\beta + 2\beta) = (1-\psi)K \end{aligned}$$

so non-negativity of curvature is preserved by the deformation.

In order to deform continuously from the original fibration one should actually start by deforming using an intermediate $\tilde{\psi}$ which is identically zero on all of $\rho^{-1}[0, R_1]$ and equal to $\psi$ outside of some small neighbourhood of $\rho^{-1}[0, R_1]$. Then one finishes the deformation by linearly interpolating from $\tilde{\psi}$ to the $\psi$. This ensures that the exact Lagrangians give a well defined exact Lagrangian boundary condition (enclosed by the $R_3$ level of $\rho$) at every stage of the deformation.

□

**Remark A.2.** In the 'flattened' region we have a trivialisation respected by $\tilde{\Omega}$, but not quite by $\tilde{\Theta}$. Namely in this region $\tilde{\Theta} = \Theta^T + dh$.

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